On parametrization of linear pseudo-differential boundary value control systems

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February 1, 2008

Abstract

The paper considers pseudo-differential boundary value control systems. The underlying operators form an algebra $\mathcal{D}$ with the help of which we are able to formulate typical boundary value control problems. The symbolic calculus gives tools to form e.g. compositions, formal adjoints, generalized right or left inverses and compatibility conditions. By a parametrizability we mean that for a given control system $\mathbf{A}u = 0$ one finds an operator $\mathbf{S}$ such that $\mathbf{A}u = 0$ if and only if $u = \mathbf{S}f$. The computation rules of $\mathcal{D}$ (or its appropriate subalgebra $\mathcal{D}'$) guarantee that in many applications $\mathbf{S}$ can be refinely analyzed or even explicitly calculated. We outline some methods of homological algebra for the study of parametrization $\mathbf{S}$. Especially the projectivity of a certain factor module (defined by the system equations) implies the parametrizability. We give some examples to illustrate our computational methods.

AMS-classification. 93B25, 93C20, 93A10, 35S99

1 Introduction

The general theory of control systems corresponding to boundary value problems for linear and nonlinear partial differential equations (PDEs) has been developed by two apparently different approaches: One has applied *functional analytic* and *algebraic* methods. In addition, *differential geometry* increasingly offers effective tools and gives geometric intuition in this field. Functional analytic methods are based on the analysis of underlying operators defined abstractly in appropriate (Banach) spaces, e.g. [1, 4, 3]. Definitions of basic concepts such as controllability, observability, stability are typically formulated using *trajectories* which have direct intuitions to the real world problems in technology and in science.

In frequence space problems (e.g. transfer function analysis) algebraic approach is more conventional [23, 21] but in state space problems algebraic study of control systems is quite recent method. The algebraic methods analyze *algebraic structures* such as structures of certain modules, exact sequences, compatibility conditions and one-sided inverses. In these approaches one mixes the input, state and output variables (which are called control variables) and studies only the structural properties of this mixed system (e.g. [24, 20, 5, 6]). In the certain linear cases one is able e.g. to apply the related (differential) algebraic modules to determine whether the system is controllable or not. The algebraic methods have often less intuitive connections to real problems.

Here we consider *parametrizability* of certain linear control systems which means that for a given control system $\mathbf{A}u = 0$ one finds an operator $\mathbf{S}$ such that $\mathbf{A}u = 0$ if and only if $u = \mathbf{S}f$. The operators $\mathbf{A}$ and $\mathbf{S}$ are built on the operators of $\mathcal{D}$ where $\mathcal{D}$ is the chosen algebra of operators. Parametrizability is a kind of structural (internal) property of the control system. The concept

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is related to the differential *flatness* of (ordinary differential) control systems ([5, 6, 2, 12]). The "parameter f" is in the role of flat output. f is *endogeneous* if it can be conversely expressed with the help of u by $f = \mathbf{Q}u$ where $\mathbf{Q}$ is built on the operators of $\mathcal{D}$. In this case the system can be considered to be "flat". In some cases parametrizability is equivalent to the *torsion freeness* of certain structural modules ([20]).

By now the structural study of state space problems, especially from the algebraic point of view, is mainly developed without boundary values. Some attempts nevertheless can be found in literature [6, 19, 15]. It is clear that the structural properties related to the boundary value control problems depend also on the boundary conditions, not only the PDE system. In the following we suggest how to take into account the boundary conditions in parametrization. We use very general formulation of boundary value systems. The operators consists of appropriate *pseudo-differential and boundary value operators*. This calculus enables, for example, the consideration of the trace operator. In addition, the calculus gives the greater freedom in the manipulation of e.g. the compositions, adjoints, compatbility conditions and left/right inverses. We express our formalism tailored for the boundary value problems related to certain partial differential equations. Besides boundary value control systems for PDEs, our approach enables also e.g. the consideration of delay systems.

Firstly we choose an appropriate *algebra* $\mathcal{D}$ of pseudo-differential and boundary value operators. This algebra originated in [14] and afterwards it has been enlarged ([7, 9]). The algebra gives very natural frame to calculate with PDEs and related boundary value operators. We generalize the definition of intrinsic parametrization (the word "intrinsic" refers to the situation where state, input and output variables are mixed) for the control systems which are corresponding to this algebra and we give some preliminary tools to study and find the parametrization. Compared with the earlier differential algebraic methods we loose certain structures such as differential fields and the derivation rules typical in the context of partial differential operators. In our case the modules are not generally over entire rings (integral domains). This follows e.g. because of the ring $C^\infty(\overline{G} \times \Delta)$ is not entire: We have $v_1 v_2 = 0$ for any nonzero functions whose supports are disjoint. Furthermore $\mathcal{D}$ as the ring of matrices is not entire.

In section 4 we formulate some preliminary ideas to study the parametrizability by certain homological algebraic methods. We define (more general) $\mathcal{D}'$-parametrizability concept, where $\mathcal{D}'$ is a *subalgebra with unity of* $\mathcal{D}$. We don't try to make clear here relations between (various variants of) controllability and parametrizability. The definition of structural controllability for the boundary value control systems (under consideration) can be potentially founded on the use of torsion freeness concept which can be generalized also for nonintegral domains. Because the underlying module is not generally over an entire ring the application of torsion concept is slightly more complicated than in the conventional case.

One aim of this paper is to show that the applied algebra gives tools to construct the operators needed in parametrization. Parametrizabilty gives potential methods for practical controller design since from $u = \mathbf{S}f$ it follows that we can express, say the input variables $c = (c_1, ..., c_p)$, the state variables $v = (v_1, ..., v_q)$ and the output variables $y = (y_1, ..., y_r)$ with the help of "(free) variables f" as

$$c = \mathbf{S}_1 f,\ v = \mathbf{S}_2 f,\ y = \mathbf{S}_3 f. \tag{1}$$

Here the operators $\mathbf{S}_j$ are constitued of the operators lying in the algebra $\mathcal{D}$ and so we can calculate $c,\ v,\ y$ using computational rules of $\mathcal{D}$. For example, in the output tracking problem one seeks the "parameter" $f = f^*$ such that (at least optimally) $y^* = \mathbf{S}_3 f^*$ where y^* is a given reference output. The required input is simply $c = c^* = \mathbf{S}_1 f^*$. This kind of "open controls" are important in some problems of modern technology (in molecular physics, for example). The other potential field of applications is the optimal control problems. Roughly speaking this option can be described as follows: If we have an optimal control problem $\min_u F(u)$ under the contraint $\mathbf{A}u = 0$, we can transform it to unconstrained problem $\min_f F(\mathbf{S}f)$.

1.1 Basic notations

Let G be an open bounded set in $\mathbf{R}^n$ and let Δ be an interval in $\mathbf{R}$. We assume that the closure $\overline{G}$ is a smooth differentiable manifold with boundary. Furthermore we suppose that the boundary ∂G is orientable and that the unit normal vector $\nu = \nu(x),\ x \in \partial G$ is pointing "outwardly" on ∂G.

The spaces $C^\infty(\overline{G})$, $C^\infty(\partial G)$, $C^\infty(\overline{G}\times\Delta)$ and $C^\infty(\partial G\times\Delta)$ are correspondingly the collections of smooth functions $\overline{G}\to\mathbf{R}$, $\partial G\to\mathbf{R}$, $\overline{G}\times\Delta\to\mathbf{R}$ and $\partial G\times\Delta\to\mathbf{R}$. $C_c^\infty(G)$ is the space of test functions and $S(\mathbf{R}^n)$ denotes the Schwartz space of rapidly decreasing functions $\mathbf{R}^n\to\mathbf{R}$. Furthermore, we define

$$\begin{aligned} C^\infty(G\times\overline{\mathbf{R}}_+) \quad &= \quad \{\phi\in C^\infty(G\times\mathbf{R}_+) \mid \text{for which there} \\ &\qquad \text{exists a } \psi\in C^\infty(G\times\mathbf{R}) \text{ such that } \phi=\psi|_{G\times\overline{\mathbf{R}}_+}\}\end{aligned}$$

and

$$C_c^\infty(G\times\overline{\mathbf{R}}_+) \quad = \quad \{\phi\in C^\infty(G\times\overline{\mathbf{R}}_+) \mid \phi \text{ has a compact support}\}.$$

$D'(G)$ is the space of distributions that is, $D'(G)$ is the dual of $C_c^\infty(G)$.

Denote

$$\mathcal{R}^{N,m} = C^\infty(\overline{G}\times\Delta)^N \oplus C^\infty(\partial G\times\Delta)^m.$$

In addition, we denote

$$\mathcal{R} = \mathcal{R}^{1,1} = C^\infty(\overline{G}\times\Delta)\oplus C^\infty(\partial G\times\Delta).$$

Let

$$\begin{aligned}\mathcal{R}_0^{N,m} \quad &= \quad \{(v,w)\in\mathcal{R}^{N,m} \mid v \text{ and } w \text{ have compact supports} \\ &\qquad \text{in } \overline{G}\times\Delta \text{ and in } \partial G\times\Delta, \text{ respectively}\}.\end{aligned}$$

For $(v,w)\in\mathcal{R}^{N,m},\ (\phi,\psi)\in\mathcal{R}_0^{N,m}$ we can define an inner product

$$\langle (v,w),(\phi,\psi)\rangle = \sum_{j=1}^N \langle v_j,\phi_j\rangle_{L_2(G\times\Delta)} + \sum_{k=1}^m \langle w_k,\psi_k\rangle_{L_2(\partial G\times\Delta)}\,. \tag{2}$$

Let $\hat{u}$ (or $\mathcal{F}u$) denote the standard Fourier transform

$$\hat{u}(\xi) = (\mathcal{F}u)(\xi) = \int_{\mathbf{R}^n} u(x)e^{-\mathrm{i}\langle x,\xi\rangle}\mathrm{d}x \tag{3}$$

for a function u in the Lebesgue space $L_1(\mathbf{R}^n)$. The inverse Fourier transform is for appropriate functions

$$\mathcal{F}^{-1}f(x) = (2\pi)^{-n}\int_{\mathbf{R}^n}\hat{f}(\xi)e^{\mathrm{i}\langle x,\xi\rangle}\mathrm{d}\xi.$$

The ring of $m\times n$-matrices is denoted by $M(m\times n)$. The partial derivative $\frac{\partial^\alpha}{\partial x^\alpha}$ is denoted more shortly by ∂_x^α.

2 Control system

2.1 Pseudo-differential and boundary value operators

We formulate the (generalized) boundary control systems with the help of pseudo-differential and boundary value operators (e.g. [7, 9]). Let $S_\nu^m(G\times\Delta\times\mathbf{R}^{n+1})$ be a space of functions, so called anisotropic symbols, $a\in C^\infty(G\times\Delta)$ such that for any compact set $K\subset G\times\Delta,\ \alpha,\ \beta\in\mathbf{N}_0^n,\ j,l\in\mathbf{N}_0$ there exists a constant $C_{\alpha,\beta,j,l,K}>0$ such that

$$|\partial_x^\alpha\partial_t^j\partial_\xi^\beta\partial_\eta^l a(x,t,\xi,\eta)| \le C_{\alpha,\beta,j,l,K}\,\rho(\xi,\eta)^{m-|\beta|-\nu l} \text{ for all } (\xi,\eta)\in\mathbf{R}^{n+1},\ (x,t)\in K \tag{4}$$

where

$$\rho(\xi,\eta) = (1+|\xi|^2+|\eta|^{2/\nu})^{1/2}.$$

As is standard, the space of symbols $S^m_\nu(G\times\Delta\times\mathbf{R}^{n+1})$ can be equipped with the topology defined by the appropriate semi-norms. In the case where we have no boundary considerations, the (classical) pseudo-differential operator $A : C^\infty_c(G\times\Delta) \to C^\infty(G\times\Delta)$ is defined by

$$A\phi(x,t) = (2\pi)^{-(n+1)} \int_{\mathbf{R}^n}\int_{\mathbf{R}} e^{\mathrm{i}\langle (x,t),(\xi,\eta)\rangle} a(x,t,\xi,\eta)\hat{\phi}(\xi,\eta)\mathrm{d}\xi\mathrm{d}\eta. \tag{5}$$

When the boundary ∂G is included in the considerations the definition of pseudo-differential operator contains some modifications. Especially, the so called transmission condition is needed. In addition, we need several classes of other operators (boundary operators).

It is sufficient to explain the operator classes only in the local case that is, in the case when the closure $\overline{G}$ is replaced with a set $G'\times\overline{\mathbf{R}}_+$ where G' is an open subset of $\mathbf{R}^{n-1}$ and where the boundary ∂G is replaced with a set G'. The technically tedious reduction to that case is based on the partition of unity and it is standard.

Generally speaking, the symbols of "boundary operators" satisfy conditions analogous, in a sense, to the transmission condition. This provides the inclusion

$$\mathcal{A}\big(C^\infty_c(G'\times\overline{\mathbf{R}}_+\times\Delta)\oplus C^\infty_c(G'\times\Delta)\big) \subset C^\infty(G'\times\overline{\mathbf{R}}_+\times\Delta)\oplus C^\infty(G'\times\Delta)$$

for any boundary value operator $\mathcal{A}$ explained in section 2.2.1. In addition, the kernels of operators involved in a boundary value problem must have proper supports. Then

$$\mathcal{A}\big(C^\infty(G'\times\overline{\mathbf{R}}_+\times\Delta)\oplus C^\infty(G'\times\Delta)\big) \subset C^\infty(G'\times\overline{\mathbf{R}}_+\times\Delta)\oplus C^\infty(G'\times\Delta)$$

and one can compose the operators of boundary problems. In the following we explain the operators in detail.

A pseudodifferential operator (ψdo)

$$r^+A\colon C^\infty_c(G'\times\overline{\mathbf{R}}_+\times\Delta) \to D'(G'\times\mathbf{R}_+\times\Delta)$$

is defined by

$$r^+Av(x,t) = (2\pi)^{-(n+1)} r^+ \int_{\mathbf{R}^n}\int_{\mathbf{R}} e^{i\langle (x,t),(\xi,\eta)\rangle} a(x,t,\xi,\eta)\widehat{(e^+v)}(\xi,\eta)\,d\xi\,d\eta.$$

The symbol a for any compact $K\subset G'\times\overline{\mathbf{R}}_+\times\Delta$ satisfies

$$|\partial^\alpha_x\partial^j_t\partial^\beta_\xi\partial^l_\eta a(x,t,\xi,\eta)| \le C_{\alpha,j,\beta,l,K}\,\rho(\xi,\eta)^{m-|\beta|-\nu l}, \quad (x,t,\xi,\eta)\in K\times\mathbf{R}^{n+1}, \tag{6}$$

where $\alpha,\beta\in\mathbf{N}^n_0$, $j,l\in\mathbf{N}_0$. The space of symbols satisfying (6) is denoted by $S^m_\nu(G'\times\overline{\mathbf{R}}_+\times\Delta\times\mathbf{R}^{n+1})$. The number m is called the *order* of the symbol a and, simultaneously, the order of ψdo r^+A. Above r^+ refers to the restriction operator $r^+f = f|_{G'\times\mathbf{R}_+\times\Delta}$ and e^+ refers to the extension by zero from $G'\times\overline{\mathbf{R}}_+\times\Delta$ on $G'\times\mathbf{R}\times\Delta$.

The symbol a satisfies the *transmission condition*, if the expansion

$$\partial^\gamma_{x_n} a(x,t,\xi,\eta)\big|_{x_n=0} = \sum_{p=0}^{m}\alpha_{\gamma p}(x',t,\xi',\eta)\xi_n^p + \sum_{k=-\infty}^{\infty} a_{\gamma k}(x',t,\xi',\eta)\frac{\big(\rho(\xi',\eta)-i\xi_n\big)^k}{\big(\rho(\xi',\eta)+i\xi_n\big)^{k+1}} \tag{7}$$

holds for all $\gamma\in\mathbf{N}_0$ while $\alpha_{\gamma p}\in S^{m-p}_\nu(G'\times\Delta\times\mathbf{R}^n)$, $a_{\gamma k}$ is rapidly decreasing sequence in $S^{m+1}_\nu(G'\times\Delta\times\mathbf{R}^n)$ (that is, for any semi-norm p on $S^{m+1}_\nu(G'\times\Delta\times\mathbf{R}^n)$ and any $N\in\mathbf{N}$ there exists a constant $c_{N,p}$ such that $p(a_{\gamma k})\le c_{N,p}(1+k)^{-N}$) and $\rho(\xi',\eta) = (1+|\xi'|^2+|\eta|^{2/\nu})^{1/2}$. The space of symbols of order m satisfying the transmission condition will be denoted by $\mathbf{U}^m_\nu$.

Assertion 1. *If a ψdo r^+A satisfies the transmission condition then $r^+Av\in C^\infty(G'\times\overline{\mathbf{R}}_+\times\Delta)$ for any $v\in C^\infty_c(G'\times\overline{\mathbf{R}}_+\times\Delta)$.*

The consideration of Assertion 1 can be reduced to the one-dimensional case in the same way as in [14], Theorem 2.9. Partial differential operators are typical examples of pseudo-differentials operators.

A potential operator. A function $k \in C^\infty(G' \times \overline{\mathbf{R}}_+ \times \Delta \times \mathbf{R}^{n+1})$ is called a potential symbol of order m if

$$k(x,t,\xi',\xi_n,\eta) = \sum_{k=0}^{\infty} a_k(x,t,\xi',\eta)\big(\rho(\xi',\eta) - i\xi_n\big)^k \big(\rho(\xi',\eta) + i\xi_n\big)^{-(k+1)}, \tag{8}$$

where a_k is a rapidly decreasing sequence in the space $S_\nu^{m+1}(G' \times \overline{\mathbf{R}}_+ \times \Delta \times \mathbf{R}^n)$ consisting of functions subjected to (6) with $\rho(\xi,\eta)$ replaced by $\rho(\xi',\eta)$. The space of potential symbols of order m is denoted by $\mathbf{K}_\nu^m$.

A potential operator is defined by

$$Kw(x,t) = (2\pi)^{(n+1)} \int_{\mathbf{R}^n}\int_{\mathbf{R}} e^{i\langle (x,t),(\xi,\eta)\rangle} k(x,t,\xi',\xi_n,\eta)\hat{w}(\xi',\eta)\, d\xi\, d\eta$$

for $w \in C_c^\infty(G' \times \Delta)$. It is obvious that $K\big(C_c^\infty(G' \times \Delta)\big) \subset C^\infty(G' \times \overline{\mathbf{R}}_+ \times \Delta)$. The (classical) solution operators of certain partial differential boundary value problems contain integral terms

$$Kg(x,t) = \int_{\partial G \times \Delta} G(x,t,y',\tau)g(y',\tau)\mathrm{d}\sigma\mathrm{d}\tau \tag{9}$$

which are examples of potential operators. The other examples are the adjoints of the below defined trace operators which are often potential operators.

A trace operator. A function $t \in C^\infty(G' \times \Delta \times \mathbf{R}^{n+1})$ is called a trace symbol of *order m* and *class d* if

$$t(x',t,\xi',\xi_n,\eta) = \sum_{p=0}^{d-1} \alpha_p(x',t,\xi',\eta)\xi_n^p + \sum_{k=0}^{\infty} a_k(x',t,\xi',\eta)\frac{\big(\rho(\xi',\eta) + i\xi_n\big)^k}{\big(\rho(\xi',\eta) - i\xi_n\big)^{k+1}} \tag{10}$$

where $\alpha_p \in S_\nu^{m-p}(G' \times \Delta \times \mathbf{R}^n)$ and a_k is a rapidly decreasing sequence in the space $S_\nu^{m+1}(G' \times \Delta \times \mathbf{R}^n)$. The space of trace symbols of order m and class d is denoted by $\mathbf{T}_\nu^{m,d}$.

A trace operator is defined by

$$Tv(x',t) = (2\pi)^{-n} \int_{\mathbf{R}^{n-1}}\int_{\mathbf{R}} e^{i\langle (x',t),(\xi',\eta)\rangle} \int^{+} t(x',t,\xi',\xi_n,\eta)(\widehat{e^+ v})(\xi,\eta)\, d\xi_n\, d\xi'\, d\eta$$

for $v \in C_c^\infty(G' \times \overline{\mathbf{R}}_+ \times \Delta)$. Here $\int^+ f d\xi_n$ is the curve integral $\int_\gamma f(z)\mathrm{d}z$ where γ is "a path rounding the upper complex half plane $\mathrm{Im} z > 0$ in the counterclockwise direction" that is, $\int^+ f d\xi_n = \lim_{R\to\infty} \oint_{\Gamma_R} f(z)dz$ where Γ_R is the boundary of the half ball $B(0,R) \cap \{z \in \mathbf{C} |\ \mathrm{Im} z > 0\}$. The operator T can be written in the form $Tv = \sum_{p=0}^{d-1} S_p\big(\partial_{x_n}^p v(x',0,t)\big) + T_0 v$ where S_p are ψdos with symbols in $S_\nu^{m-p}(G' \times \Delta \times \mathbf{R}^n)$ and T_0 is of class 0. It is obvious that $T\big(C_c^\infty(G' \times \overline{\mathbf{R}}_+ \times \Delta)\big) \subset C^\infty(G' \times \Delta)$. Restrictions of partial differential operators on the boundary (which usually appear in boundary conditions) are typical trace operators.

A Green operator. A function $b \in C^\infty(G' \times \overline{\mathbf{R}}_+ \times \Delta \times \mathbf{R}^{n+2})$ is called a Green symbol of *order m* and *class d* if

$$\begin{aligned} &b(x,t,\xi',\xi_n,\zeta_n,\eta) \\ &= \sum_{p=0}^{d-1} k_p(x,t,\xi',\xi_n,\eta)\zeta_n^p + \sum_{j,l=0}^{\infty} a_{jl}(x',t,\xi',\eta)\frac{(\rho(\xi',\eta) - i\xi_n)^j}{(\rho(\xi',\eta) + i\xi_n)^{j+1}}\frac{(\rho(\xi',\eta) + i\zeta_n)^l}{(\rho(\xi',\eta) - i\zeta_n)^{l+1}}, \end{aligned} \tag{11}$$

where $k_p \in \mathbf{K}_\nu^{m-p}$ and a_{jl} is a rapidly decreasing double sequence in the space $S_\nu^{m+2}(G' \times \Delta \times \mathbf{R}^n)$. The space of Green symbols of order m and class d is denoted by $\mathbf{B}_\nu^{m,d}$.

A Green operator is defined by

$$Bv(x,t) = (2\pi)^{-n-1}\int_{\mathbf{R}^n}\int_{\mathbf{R}} e^{i\langle (x,t),(\xi,\eta)\rangle}\int^+ b(x,t,\xi',\xi_n,\zeta_n,\eta)\widehat{(e^+v)}(\xi',\zeta_n,\eta)\,d\zeta_n\,d\xi\,d\eta$$

for $v\in C_c^\infty(G'\times\overline{\mathbf{R}}_+\times\Delta)$. The operator B can be written in the form $Bv = \sum_{p=0}^{d-1} K_p\big(\partial_{x_n}^p v(x',0,t)\big) + B_0 v$ where K_p are potential operators with symbols in $\mathbf{K}_\nu^{m-p}$ and B_0 is of class 0. It is obvious that $B\big(C_c^\infty(G'\times\overline{\mathbf{R}}_+\times\Delta)\big)\subset C^\infty(G'\times\overline{\mathbf{R}}_+\times\Delta)$. Typical sources of singular Green operators are the compositions of potential and trace operators. They are also born in the compositions of truncated pseudo-differential operators. Finally they appear e.g. in many solution operators related to the usual boundary value problems.

A pseudodifferential operator on the boundary

$$Q\colon C_c^\infty(G'\times\Delta)\to C^\infty(G'\times\Delta)$$

is defined by

$$Qw(x',t) = (2\pi)^{-n}\int_{\mathbf{R}^{n-1}}\int_{\mathbf{R}} e^{i\langle (x',t),(\xi',\eta)\rangle} q(x',t,\xi',\eta)\hat{w}(\xi',\eta)\mathrm{d}\xi'\mathrm{d}\eta.$$

with the symbol $q\in S_\nu^m(G'\times\Delta\times\mathbf{R}^n)$. Partial differential operators on the boundary (manifold) form an example of these operators.

In practice the operator r^+A is often a partial differential operator. The operator T is a natural generalization of the usual partial differential trace operator appeared in classical theory of boundary value problems. The operators K, B and Q are needed e.g. in the consideration of the compositions, inverses and adjoints. The operators $T,\ K,\ B,\ Q$ are called *boundary operators.* Together with these operators one can formulate very rich variety of boundary value problems. This more general formulation also gives more symmetry in the calculus of adjoints and compositions.

Example 1. *Let $L(D) = a\partial_x^2 + b\partial_x + c$ be a PDO with constant coefficients and let $d_1,\ d_2$ be constants (in this example we have no time variable). Furthermore, let $G =]0,1[\subset\mathbf{R}$. Then under relevant assumptions the solution v_1 for the system*

$$L(D)v_1 = q(x)v_2 \tag{12}$$

$$\partial_x v_1(0) + d_1 v_1(0) = 0,\ \ \partial_x v_1(1) + d_2 v_1(1) = 0 \tag{13}$$

is formally given by ([15])

$$v = P_0(qv_2) + R_0(qv_2) \tag{14}$$

where P_0 is a pseudo-differential operator

$$P_0\phi(x) = (2\pi)^{-1}\int_{\mathbf{R}} e^{\mathrm{i}\langle x,\xi\rangle}\frac{1}{L(\xi)}\hat{\phi}(\xi)\mathrm{d}\xi \tag{15}$$

and R_0 is a singular Green operator (note that in one-dimensional case the integration reduces to summation)

$$\begin{aligned} R_0\phi(x) \quad &= \quad -\frac{g_1(x,0)}{W(0)}[(\frac{1}{2} + g_2(0,0))P_0\phi(1) - g_2(0,1)P_0\phi(0)] \\ &\qquad -\frac{g_2(x,0)}{W(0)}[g_1(0,0))P_0\phi(1) - (\frac{1}{2} + g_1(0,1))P_0\phi(0)]. \end{aligned} \tag{16}$$

Here $L(\xi) = -a\xi^2 + b\mathrm{i}\xi - c$ is the symbol of $L(D)$ and

$$g_k(x,\lambda) = (2\pi)^{-1}\mathbf{pv}\int_{\mathbf{R}}\left[e^{\mathrm{i}\xi x}\frac{g_k(\xi)}{\lambda - L(\xi)}\mathrm{d}\xi\right],\ \ \lambda = 0,1,\ \ k = 1,2,$$

$$W(0) = g_1(0,0)g_2(1,0) - (1 + g_1(1,0))(1 + g_2(0,0)),$$

where

$$g_1(\xi) = (ad_2 - b - a\mathrm{i}\xi)e^{-\mathrm{i}\xi},\ \ g_2(\xi) = b - ad_1 + a\mathrm{i}\xi.$$

Here **pv** *denotes that the integral is taken in the sense of principal value.*

2.2 Computation rules

We survey some basic computational properties of the above defined operators.

2.2.1 Operator algebra

A boundary value operator $\mathcal{A} = \begin{pmatrix} r^+A + B & K \\ T & Q \end{pmatrix}$ is called *proper* if the kernels of all operators r^+A, B, K, T, Q have proper supports. The space of proper boundary value operators is denoted by $\mathcal{D}$

Assertion 2. *If $\mathcal{A}$ and $\mathcal{B}$ are in $\mathcal{D}$, then $\mathcal{AB} := \mathcal{A} \circ \mathcal{B} \in \mathcal{D}$.*

To verify this assertion one can modify the arguments used in [7] for the case of parameter dependent boundary value problems.

We say that the operator $\mathcal{A} = \begin{pmatrix} r^+A + B & K \\ T & Q \end{pmatrix}$ is of class $\mathcal{D}_\nu^{m,d}$,

$$m = \begin{pmatrix} m_1, m_2 & m_3 \\ m_4 & m_5 \end{pmatrix}, \qquad d = \begin{pmatrix} d_2 \\ d_4 \end{pmatrix}$$

when

the symbol of r^+A is in the class $\mathbf{U}_\nu^{m_1}$,
the symbol of B is in the class $\mathbf{B}_\nu^{m_2,d_2}$,
the symbol of K is in the class $\mathbf{K}_\nu^{m_3}$,
the symbol of T is in the class $\mathbf{T}_\nu^{m_4,d_4}$,
the symbol of Q is in the class $S_\nu^{m_5}$.

The assertion 2 implies that proper boundary value operators of arbitrary order m and class d form an *algebra* $\mathcal{D}$ with respect to standard addition and composition of operators. In addition, from the above definitions and assertions it follows that the spaces $\mathcal{D}$ and $C^\infty(G' \times \overline{\mathbf{R}}_+ \times \Delta) \oplus C^\infty(G' \times \Delta)$ are *$\mathcal{D}$-modules.* The order and the class of $\mathcal{A}B$ can be calculated with the help of orders and classes of $\mathcal{A}$ and $\mathcal{B}$.

In the global setting we have operators

$$r^+A : C^\infty(\overline{G} \times \Delta) \to C^\infty(\overline{G} \times \Delta)$$

$$K : C^\infty(\partial G \times \Delta) \to C^\infty(\overline{G} \times \Delta)$$
$$T : C^\infty(\overline{G} \times \Delta) \to C^\infty(\partial G \times \Delta)$$
$$B : C^\infty(\overline{G} \times \Delta) \to C^\infty(\overline{G} \times \Delta)$$
$$Q : C^\infty(\partial G \times \Delta) \to C^\infty(\partial G \times \Delta)$$

and

$$\mathcal{A} : \mathcal{R} \to \mathcal{R}$$

where $\mathcal{R} = C^\infty(\overline{G} \times \Delta) \oplus C^\infty(\partial G \times \Delta)$. In addition $\mathcal{R}$ is a $\mathcal{D}$-module.

2.2.2 Formal adjoint

As mentioned above in $\mathcal{R} = C^\infty(\overline{G} \times \Delta) \oplus C^\infty(\partial G \times \Delta)$ we use the inner product (e.g. for functions $U_1 \in \mathcal{R},\ U_2 \in \mathcal{R}_0$)

$$\langle U_1, U_2 \rangle = \langle v_1, v_2 \rangle_{L_2(G \times \Delta)} + \langle w_1, w_2 \rangle_{L_2(\partial G \times \Delta)}$$

where

$$U_k = (v_k, w_k),\ k = 1, 2.$$

For any boundary value operator $\mathcal{A} \in \mathcal{D}$ there exists an operator

$$\mathcal{B} : C_c^\infty(\overline{G} \times \Delta) \oplus C_c^\infty(\partial G \times \Delta) \to D'(G \times \Delta) \oplus D'(\partial G \times \Delta)$$

such that $(\mathcal{A}U, V) = (U, \mathcal{B}V)$ where $(\cdot,\cdot)$ is the standard duality form on

$$(C_c^\infty(\overline{G}\times\Delta)\oplus C_c^\infty(\partial G\times\Delta))\times(D'(G\times\Delta)\oplus D'(\partial G\times\Delta)).$$

If the operator $\mathcal{B}$ maps the space $\mathcal{R}$ into itself we will say that formal adjoint to $\mathcal{A}$ exists and will denote the operator $\mathcal{B}$ by $\mathcal{A}^*$. When the operator $\mathcal{A}^* : \mathcal{R}\to\mathcal{R}$ exists we have

$$\langle V, \mathcal{A}U\rangle = \langle \mathcal{A}^*V, U\rangle \tag{17}$$

for $V, U \in \mathcal{R}_0$.

Assertion 3. *The adjoint operator $\mathcal{A}^*$ exists and is in the algebra $\mathcal{D}$ if and only if the operator $\mathcal{A} = \begin{pmatrix} r^+A + B & K \\ T & Q\end{pmatrix}$ is of class 0 (i.e., the class of operators B and T equals 0) and in the transmission condition (7) for the symbol of r^+A all $\alpha_{\gamma p}$, except $\alpha_{\gamma 0}$, are equal to 0. In the case where $\mathcal{A}^*$ exists then there exists $\mathcal{A}^{**}$ and $\mathcal{A}^{**} = \mathcal{A}$.*

We omit here the proof of Assertion 3 but we remark that analogous results and considerations can be found in the monograph [7].

We find that for an operator $\mathcal{A}\in\mathcal{D}$ the formal adjoint $\mathcal{A}^*$, generally speaking, does not exist. However, one can consider subset $\mathcal{D}_1\subset\mathcal{D}$ that consists of boundary value operators satisfying conditions of the Assertion 3. Then $\mathcal{D}_1$ is an algebra and for any $\mathcal{A}\in\mathcal{D}_1$ there exists the formal adjoint $\mathcal{A}^*$ in $\mathcal{D}_1$. If we fix $m_0 = \begin{pmatrix} 0,-1 & -1/2 \\ -1/2 & 0\end{pmatrix}$ and $d_0 = \begin{pmatrix} 0 \\ 0\end{pmatrix}$, then the subalgebra $\mathcal{D}_\nu^{m_0,d_0}\subset\mathcal{D}_1$ possesses the same properties as $\mathcal{D}_1$.

Example 2. *Let $G =]0,1[\subset \mathbf{R}$ and*

$$Tv = d(\partial)v|_{\partial G}$$

where $d(\partial)$ is a pseudo-differential operator

$$d(\partial)v(x) = (2\pi)^{-1}\int_{\mathbf{R}} e^{\mathrm{i}\xi x} d(\xi)\widehat{e^+v}(\xi)\mathrm{d}\xi,$$

such that its symbol $d(\xi)$ satisfies

$$|d(\xi)| \le C\frac{1}{(1+|\xi|)^\kappa},\ \ \kappa > 1. \tag{18}$$

Then we have for $v\in C^\infty(\overline{G}),\ g\in C^\infty(\partial G)$

$$\begin{aligned}
&\langle Tv, g\rangle_{L_2(\partial G)} = (Tv)(0)\overline{g(0)} + (Tv)(1)\overline{g(1)}\\
&= (2\pi)^{-1}\overline{g(0)}\int_{\mathbf{R}} d(\xi)\widehat{e^+v}(\xi)\mathrm{d}\xi + (2\pi)^{-1}\overline{g(1)}\int_{\mathbf{R}} e^{\mathrm{i}\xi} d(\xi)\widehat{e^+v}(\xi)\mathrm{d}\xi\\
&= (2\pi)^{-1}\left(\int_G\left(\overline{g(0)}\int_{\mathbf{R}} e^{-\mathrm{i}\xi x}d(\xi)\mathrm{d}\xi\right)v(x)\mathrm{d}x + \int_G\left(\overline{g(1)}\int_{\mathbf{R}} e^{\mathrm{i}\xi(-x+1)}d(\xi)\mathrm{d}\xi\right)v(x)\mathrm{d}x\right)\\
&= \langle v, T^*g\rangle_{L_2(G)}
\end{aligned}$$

where

$$T^*g(x) = (2\pi)^{-1}\left(g(0)\int_{\mathbf{R}} e^{-\mathrm{i}\xi x}\overline{d(\xi)}\mathrm{d}\xi + g(1)\int_{\mathbf{R}} e^{\mathrm{i}\xi(-x+1)}\overline{d(\xi)}\mathrm{d}\xi\right). \tag{19}$$

We see that T^* is a potential-type operator, but it does not generally possess the transmission property. Consideration of operators like (19) can be found in [25].

2.2.3 Order reducing operators

The so called *order reduction* can be used to help the forming of adjoints in parametrization processes.

Let $\mathcal{A}$ be a boundary value operator of order $m = \begin{pmatrix} m_1, m_2 & m_3 \\ m_4 & m_5 \end{pmatrix}$ and class $d = \begin{pmatrix} d_2 \\ d_4 \end{pmatrix}$. If $T = \begin{pmatrix} r^+T_1 & 0 \\ 0 & T_2 \end{pmatrix}$ while $\operatorname{ord} r^+T_1 = -N_1$ (that is, the order of r^+T_1 is $-N_1$) and $\operatorname{ord} T_2 = -N_2$, $N_1,\ N_2 \in \mathbf{N}$ then the operator $\mathcal{A} \circ T$ is of order $\begin{pmatrix} m_1 - N_1, m_2 - N_1 & m_3 - N_2 \\ m_4 - N_1 & m_5 - N_2 \end{pmatrix}$ and class $\begin{pmatrix} d_2 - N_1 \\ d_4 - N_1 \end{pmatrix}$. Note that the adjoint operator $(\mathcal{A} \circ T)^*$ exists if $N_1 > \max(m_1, d_2, d_4)$.

2.3 System equations

In the following we denote more shortly by $\mathcal{A} = \begin{pmatrix} \mathcal{A}_1 & \mathcal{A}_2 \\ \mathcal{A}_3 & \mathcal{A}_4 \end{pmatrix}$ a typical element $\mathcal{A} = \begin{pmatrix} r^+A + B & K \\ T & Q \end{pmatrix}$ of $\mathcal{D}$.

Let $v_1, ..., v_N$ and $w_1, ..., w_m$ be indeterminates in $C^\infty(\overline{G} \times \Delta)$ and in $C^\infty(\partial G \times \Delta)$, respectively. Furthermore, let $(\mathcal{A}_{1ij}),\ (\mathcal{A}_{2ik}),\ (\mathcal{A}_{3lj}),\ (\mathcal{A}_{4lk})$ be operator matrices where $\mathcal{A}_{1ij}$ are of type $r^+A_{ij} + B_{ij}$, $\mathcal{A}_{2ik}$ are of type K_{ik}, $\mathcal{A}_{3lj}$ are of type T_{lj} and $\mathcal{A}_{4lk}$ are of type Q_{lk}. We consider the control system

$$\begin{aligned} \sum_{j=1}^{N} \mathcal{A}_{1ij} v_j + \sum_{k=1}^{m} \mathcal{A}_{2ik} w_k &= 0, \quad i = 1, ..., N_1, \\ \sum_{j=1}^{N} \mathcal{A}_{3lj} v_j + \sum_{k=1}^{m} \mathcal{A}_{4lk} w_k &= 0, \quad l = 1, ..., m_1. \end{aligned} \tag{20}$$

The state, input and output variables are not separated.

The system (20) can be given in a matrix form as follows. Let $(\mathcal{A}_{1ij}) \in M(N_1 \times N)$, $(\mathcal{A}_{2ik}) \in M(N_1 \times m)$, $(\mathcal{A}_{3lj}) \in M(m_1 \times N)$ and $(\mathcal{A}_{4lk}) \in M(m_1 \times m)$ be the matrices corresponding to the system (20). Denote $u = \begin{pmatrix} v \\ w \end{pmatrix}$ where $v = \begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix}$, $w = \begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix}$ and denote

$$\mathbf{A} = \begin{pmatrix} (\mathcal{A}_{1ij}) & (\mathcal{A}_{2ik}) \\ (\mathcal{A}_{3lj}) & (\mathcal{A}_{4lk}) \end{pmatrix} \in M((N_1 + m_1) \times (N + m)). \tag{21}$$

Then the system (20) is equivalent to the equation

$$\mathbf{A}u = 0. \tag{22}$$

The operator $\mathbf{A}$ is a linear operator $\mathcal{R}^{N,m} \to \mathcal{R}^{N_1,m_1}$. We say the the operator $\mathbf{A}$ is in $\mathcal{D}(N_1 + m_1, N + m)$ when it is of the form (21). This notation uniquelly indicates the types of submatrices. Using the standard *Frechet space* topologies in $C^\infty(\overline{G} \times \Delta)$ and in $C^\infty(\partial G \times \Delta)$ one has
Assertion 4. *The linear operator* $\mathbf{A} : \mathcal{R}^{N,m} \to \mathcal{R}^{N_1,m_1}$ *is continuous.*

Example 3. *In this example we outline how to choose the above operators in the case of second order linear PDEs. Consider a linear coupled system of partial differential equations*

$$\sum_{j=1}^{N} A_{ij}(x, t, \partial) v_j = 0, \tag{23}$$

for $i = 1, ..., N_1$*, where* $A_{ij}(x,t,\partial)$ *are, for example, the second order operators*

$$A_{ij}(x,t,\partial) = \sum_{|\sigma|\le 2} a_\sigma^{ij}(x,t)\partial^\sigma,$$

where

$$\partial = \left(\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}, \frac{\partial}{\partial t}\right) =: (\partial_x, \partial_t).$$

Denote $\mathcal{A}_{1ij}(x,t,\partial) = A_{ij}(x,t,\partial)$.

We assume that the solution satisfies the following homogeneous boundary conditions

$$\left[\sum_{j=1}^{N} d_{lj}(x,t,\partial_x)v_j\right]|_{\partial G\times\Delta} = 0, \;\; l = 1, ..., m_1 \tag{24}$$

where $d_{lj}(x,t,\partial_x)$ *are first order partial differential operators*

$$d_{lj}(x,t,\partial_x) = \sum_{k=1}^{n} d_{ljk}(x,t)\frac{\partial}{\partial x_k} + d_{lj0}(x,t).$$

In this example we choose the operator matrix $(\mathcal{A}_{1ij})$ *such that*

$$(\mathcal{A}_{1ij}) = \begin{pmatrix} r^+A_{11}(x,t,\partial) & \cdots & r^+A_{1N}(x,t,\partial)) \\ \vdots & & \\ r^+A_{N_11}(x,t,\partial)) & \cdots & r^+A_{N_1N}(x,t,\partial)) \end{pmatrix},$$

where $r^+f := f|_{G\times\Delta}$ *is (as above) the restriction operator on* $G\times\Delta$*. The boundary operators* B_{ij}*,* K_{ik}*,* Q_{lk} *are zero operators and the operators* $\mathcal{A}_{3lj}$ *corresponding to the boundary operators* T_{lj} *are defined by*

$$(\mathcal{A}_{3lj}) = \begin{pmatrix} r'd_{11}(x,t,\partial_x) & \cdots & r'd_{1N}(x,t,\partial_x) \\ \vdots & & \\ r'd_{m_11}(x,t,\partial_x) & \cdots & r'd_{m_1N}(x,t,\partial_x) \end{pmatrix} \tag{25}$$

where $r'f := f|_{\partial G\times\Delta}$ *is the* restriction operator *on* $\partial G\times\Delta$*. As a conclusion we find that*

$$\mathbf{A} = \begin{pmatrix} (\mathcal{A}_{1ij}) & 0 \\ (\mathcal{A}_{3lj}) & 0 \end{pmatrix}. \tag{26}$$

Remark 1. *When we choose* $d_{lj}(x,t,\partial_x) = 0,\; l = 1, ..., m_1,\; j = 1, ..., N$ *the function* v_j *does not satisfy any boundary condition on* ∂G*.*

Example 4. *Consider the delay system*

$$\frac{dv}{dt} = A_0v + \sum_{k=1}^{p} A_kv(t-h_k) + B_0V \tag{27}$$

where $0 < h_1 < \cdots < h_p$*,* $v(t) \in \mathbf{C}^N,\; V(t) \in \mathbf{C}^q$ *and* $A_0,\; A_k \in M(N\times N),\; B_0 \in M(N\times q)$*.*

Noting that

$$f(t-h_k) = (2\pi)^{-1}\int_{\mathbf{R}} \hat{f}(\eta)e^{\mathrm{i}(t-h_k)\eta}\mathrm{d}\eta \tag{28}$$
$$= (2\pi)^{-1}\int_{\mathbf{R}} e^{-\mathrm{i}h_k\eta}\hat{f}(\eta)e^{\mathrm{i}t\eta}\mathrm{d}\eta = P_kf(t)$$

we see that the system (27) can be written in the form

$$\mathbf{A}\begin{pmatrix} v \\ V \end{pmatrix} = 0 \tag{29}$$

where

$$\mathbf{A} = \left(\tfrac{d}{dt} - A_0 - \textstyle\sum_{k=1}^{p} A_k P_k \quad -B_0 \right). \tag{30}$$

Here P_k is a pseudodiffererential operator with symbol $p_k(\eta) = e^{-\mathrm{i}h_k\eta}$.

3 Parametrization of a control system applying algebraic methods

3.1 Parametrizability of a control system

Here we consider parametrizability of the above defined boundary value control systems. The control system

$$\begin{aligned} \sum_{j=1}^{N} \mathcal{A}_{1ij} v_j + \sum_{k=1}^{m} \mathcal{A}_{2ik} w_k &= 0,\ i = 1, ..., N_1, \\ \sum_{j=1}^{N} \mathcal{A}_{3lj} v_j + \sum_{j=1}^{m} \mathcal{A}_{4lk} w_k &= 0,\ l = 1, ..., m_1, \end{aligned} \tag{31}$$

or more simply

$$\mathbf{A}u = 0 \tag{32}$$

is said to be *parametrizable*, if there exists $N',\ m' \in \mathbf{N}$ and a linear operator

$$\mathbf{S} = \begin{pmatrix} (\mathcal{S}_{1jp}) & (\mathcal{S}_{2jq}) \\ (\mathcal{S}_{3kp}) & (\mathcal{S}_{4kq}) \end{pmatrix} \in \mathcal{D}(N + m, N' + m') \tag{33}$$

such that

$$\mathbf{A}u = 0 \Leftrightarrow u = \mathbf{S}f \tag{34}$$

(the components f_m of f are $\mathcal{D}$-linear independent). The condition (34) means that the equation $\mathbf{A}u = 0$ determinates all *compatibility conditions* of the equation $\mathbf{S}f = u$.

Example 5. *Let $G \subset \mathbf{R}^2$. Denote $\partial_1 = \frac{\partial}{\partial x_1}$, $\partial_2 = \frac{\partial}{\partial x_2}$ and let I, I' be the identity mappings on $C^\infty(\overline{G})$ and $C^\infty(\partial G)$, respectively. Consider a system*

$$\partial_2 v_1 - \partial_1 v_1 - \partial_2 v_2 + v_2 = 0.$$

The system can be put into the form (the inclusion of w_1 is due to the notational convenience)

$$\begin{pmatrix} \partial_2 - \partial_1 & -\partial_2 + I & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ w_1 \end{pmatrix} = 0.$$

The system has a parametrization ([20])

$$\begin{pmatrix} v_1 \\ v_2 \\ w_1 \end{pmatrix} = \begin{pmatrix} \partial_2 - I & 0 \\ \partial_2 - \partial_1 & 0 \\ 0 & I' \end{pmatrix} \begin{pmatrix} f \\ w_1 \end{pmatrix} =: \mathbf{S}\begin{pmatrix} f \\ w_1 \end{pmatrix}. \tag{35}$$

Example 6. *Consider the control system related to Example 1*

$$L(D)v_1 = q(x)v_2 \tag{36}$$
$$\partial_x v_1(0) + d_1 v_1(0) = 0,\ \ \partial_x v_1(1) + d_2 v_1(1) = 0$$
$$y = v_1|_{\partial G}. \tag{37}$$

This system can be put into the form

$$\begin{pmatrix} L(D) & -q(x)I & 0 \\ T & 0 & 0 \\ r' & 0 & -I' \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ y \end{pmatrix} = 0. \tag{38}$$

Here I (resp. I') is the identity mapping on $C^\infty(\overline{G})$ (resp. $C^\infty(\partial G)$). T is the trace operator

$$Tv_1 = \begin{cases} \partial_x v_1(x) + d_1 v_1(x), & x = 0 \\ \partial_x v_1(x) + d_2 v_1(x), & x = 1 \end{cases}. \tag{39}$$

Due to Example 1 the system (38) has a parametrization given by

$$\begin{pmatrix} v_1 \\ v_2 \\ y \end{pmatrix} = \begin{pmatrix} (P_0 + R_0)(q(\cdot)) \\ I \\ r'(P_0 + R_0)(q(\cdot)) \end{pmatrix} v_2 \tag{40}$$

The homological algebra gives tools to study structurally the parametrizability. We describe the basic idea as follows. Suppose that $\mathcal{D}$ is an algebra of suitable operators. Let $P_1,\ P_2,\ M$ be $\mathcal{D}$-modules and let $d_1 : P_2 \to P_1$ and $d_2 : P_1 \to M$ be $\mathcal{D}$-homomorphisms. Recall that the sequence of modules

$$P_2 \xrightarrow{d_1} P_1 \xrightarrow{d_2} M \longrightarrow 0$$

is *complex* if im $d_1 \subset$ ker d_2 and it is *exact* if ker $d_2 =$ im d_1. The exactness means that $d_2 v = 0$ if and only if $v = d_1 f$ and so the solutions of system $d_2 v = 0$ can be *parametrized* by d_1. The *homology* of the above complex is the quotient ker d_2/im d_1. The homology measures how much the sequence differs from being exact. Furthermore, using certain homology groups, $\mathrm{Ext}^n(M, A)$, one can eventually specify how far a given module is e.g. from being projective. Using the groups $\mathrm{Tor}^n(M, A)$ one is able to measure how far a given module is from being flat.

3.2 System modules

As we found above, $\mathcal{D}$ is an algebra of operators. Hence it is a (left) $\mathcal{D}$-module (for basic concepts of homological algebra see e.g. [22, 10, 17, 16]). The module $\mathcal{D}$ has a unit

$$\mathcal{I} = \begin{pmatrix} r^+ I & 0 \\ 0 & I' \end{pmatrix} \tag{41}$$

where I is the identity operator $C^\infty(\overline{G} \times \Delta) \to C^\infty(\overline{G} \times \Delta)$ and I' is the identity operator $C^\infty(\partial G \times \Delta) \to C^\infty(\partial G \times \Delta)$. Hence $\mathcal{D}$ is a unitary $\mathcal{D}$-module.

We at first *choose a subalgebra* $\mathcal{D}'$ of $\mathcal{D}$ which contains an unit I. Furthermore, let $\mathcal{D}'^N$ be the direct product of modules

$$\mathcal{D}'^N = \mathcal{D}' \times \cdots \times \mathcal{D}'. \tag{42}$$

Then $\mathcal{D}'^N$ is a (left) free $\mathcal{D}'$-module whose canonical basis is

$$E_1 = (\mathcal{I}, 0, \cdots, 0),\ \ldots\ E_N = (0, \cdots, 0, \mathcal{I}). \tag{43}$$

Example 7. *The following sets of operators are subalgebras of $\mathcal{D}$*

1. $\mathcal{D}' = \{\begin{pmatrix} r^+A(D) & 0 \\ 0 & 0 \end{pmatrix} \mid A(D)$ is a PDO with constant coefficients$\}$,

2. $\mathcal{D}' = \{\begin{pmatrix} r^+A(x,t,D) & 0 \\ 0 & 0 \end{pmatrix} \mid A(x,t,D)$ is a PDO with real analytic coefficients$\}$,

3. $\mathcal{D}' = \{\begin{pmatrix} r^+A(x,t,D) & 0 \\ r'd(x,t,D) & aI' \end{pmatrix} \mid A(x,t,D),\ d(x,t,D)$ *are PDOs with* $C^\infty(\overline{G}\times\Delta)$*- coefficients,* $a \in \mathbf{R}\}$,

4. $\mathcal{D}' = \mathcal{D}_1$*, where* $\mathcal{D}_1$ *is as in subsection 2.2.2.*

Suppose that the control system

$$\begin{aligned} \sum_{j=1}^{N} \mathcal{A}_{1ij} v_j + \sum_{k=1}^{m} \mathcal{A}_{2ik} w_k &= 0,\ i = 1, ..., N_1, \\ \sum_{j=1}^{N} \mathcal{A}_{3lj} v_j + \sum_{k=1}^{m} \mathcal{A}_{4lk} w_k &= 0,\ l = 1, ..., m_1, \end{aligned} \tag{44}$$

is given. Let $\overline{N} = \max\{N, m\}$, $\overline{N}_1 = \max\{N_1, m_1\}$. The system (44) can be expressed equivalently in the form

$$\begin{pmatrix} \mathcal{L}_{11} & \cdots & \mathcal{L}_{1\overline{N}} \\ \cdots & \cdots & \cdots \\ \mathcal{L}_{\overline{N}_1 1} & \cdots & \mathcal{L}_{\overline{N}_1\overline{N}} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} v_1 \\ w_1 \end{pmatrix} \\ \vdots \\ \begin{pmatrix} v_{\overline{N}} \\ w_{\overline{N}} \end{pmatrix} \end{pmatrix} = 0. \tag{45}$$

For example, suppose that $N > m,\ N_1 > m_1$. Define

$$\mathcal{L}_{ij} = \begin{cases} \begin{pmatrix} \mathcal{A}_{1ij} & \mathcal{A}_{2ij} \\ \mathcal{A}_{3ij} & \mathcal{A}_{4ij} \end{pmatrix} & \text{for } 1 \le j \le m,\ 1 \le i \le m_1, \\ \begin{pmatrix} \mathcal{A}_{1ij} & 0 \\ \mathcal{A}_{3ij} & 0 \end{pmatrix} & \text{for } m+1 \le j \le N,\ 1 \le i \le m_1, \\ \begin{pmatrix} \mathcal{A}_{1ij} & \mathcal{A}_{2ij} \\ 0 & 0 \end{pmatrix} & \text{for } 1 \le j \le m,\ m_1+1 \le i \le N_1, \\ \begin{pmatrix} \mathcal{A}_{1ij} & 0 \\ 0 & 0 \end{pmatrix} & \text{for } m+1 \le j \le N,\ m_1+1 \le i \le N_1. \end{cases} \tag{46}$$

Similarly we can reformulate the other cases.

Example 8. *The system*

$$\mathbf{A}u = \begin{pmatrix} \mathcal{A}_{111} & \mathcal{A}_{112} & \mathcal{A}_{113} & \mathcal{A}_{211} & \mathcal{A}_{212} \\ \mathcal{A}_{121} & \mathcal{A}_{122} & \mathcal{A}_{123} & \mathcal{A}_{221} & \mathcal{A}_{222} \\ \mathcal{A}_{311} & \mathcal{A}_{312} & \mathcal{A}_{313} & \mathcal{A}_{411} & \mathcal{A}_{412} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ w_1 \\ w_2 \end{pmatrix} = 0$$

can be put into the form

$$\mathcal{L} \begin{pmatrix} \begin{pmatrix} v_1 \\ w_1 \end{pmatrix} \\ \begin{pmatrix} v_2 \\ w_2 \end{pmatrix} \\ \begin{pmatrix} v_3 \\ w_3 \end{pmatrix} \end{pmatrix} = 0,$$

where

$$\mathcal{L} = \begin{pmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \end{pmatrix} = \left(\begin{pmatrix} \mathcal{A}_{111} & \mathcal{A}_{211} \\ \mathcal{A}_{311} & \mathcal{A}_{411} \end{pmatrix} \begin{pmatrix} \mathcal{A}_{112} & \mathcal{A}_{212} \\ \mathcal{A}_{312} & \mathcal{A}_{412} \end{pmatrix} \begin{pmatrix} \mathcal{A}_{113} & 0 \\ \mathcal{A}_{313} & 0 \end{pmatrix} \atop \begin{pmatrix} \mathcal{A}_{121} & \mathcal{A}_{221} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{A}_{122} & \mathcal{A}_{222} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{A}_{123} & 0 \\ 0 & 0 \end{pmatrix} \right).$$

Let $\mathcal{D}'(m \times n)$ be the set of $m \times n$-matrices whose elements are in $\mathcal{D}'$. Denote $\mathcal{L} = (\mathcal{L}_{ij}) \in \mathcal{D}'(\overline{N}_1 \times \overline{N})$ and denote $u_j = \begin{pmatrix} v_j \\ w_j \end{pmatrix}$, $j = 1, ..., \overline{N}$. Let $u = \begin{pmatrix} u_1 \\ \vdots \\ u_{\overline{N}} \end{pmatrix}$. Then the system (45) is

$$\mathcal{L}u = 0. \tag{47}$$

Define a mapping $.\mathcal{L} : \mathcal{D}'^{\overline{N}_1} \to \mathcal{D}'^{\overline{N}}$ by

$$.\mathcal{L}D = D\mathcal{L} = \begin{pmatrix} D_1 & \cdots & D_{\overline{N}_1} \end{pmatrix} \begin{pmatrix} \mathcal{L}_{11} & \cdots & \mathcal{L}_{1\overline{N}} \\ \cdots & \cdots & \cdots \\ \mathcal{L}_{\overline{N}_1 1} & \cdots & \mathcal{L}_{\overline{N}_1 \overline{N}} \end{pmatrix} \tag{48}$$

for $D = \begin{pmatrix} D_1 & \cdots & D_{\overline{N}_1} \end{pmatrix} \in \mathcal{D}'^{\overline{N}_1}$. Then we find that $.\mathcal{L}$ is a (left) $\mathcal{D}'$-homomorphism that is,

$$.\mathcal{L}(L_1 D + L_2 D') = L_1.\mathcal{L}D + L_2.\mathcal{L}D'$$

for $L_1,\ L_2 \in \mathcal{D}',\ D,\ D' \in \mathcal{D}'^{\overline{N}_1}$. The image $\mathrm{im}(.\mathcal{L})$ is exactly $\mathcal{D}'^{\overline{N}_1}\mathcal{L}$ and it is a submodule of $\mathcal{D}'^{\overline{N}}$. Hence we are able to define the (left) factor module

$$M = \mathrm{coker}(.\mathcal{L}) = \mathcal{D}'^{\overline{N}} / \mathcal{D}'^{\overline{N}_1}\mathcal{L}. \tag{49}$$

The elements of M are of the form $[D] = D + \mathcal{D}'^{\overline{N}_1}\mathcal{L}$. Let $\pi : \mathcal{D}'^{\overline{N}} \to M$ be the canonical surjection

$$\pi D = [D]. \tag{50}$$

π is a surjective $\mathcal{D}'$-homomorphism and $\ker \pi = \mathcal{D}'^{\overline{N}_1}\mathcal{L}$ and so we have an exact sequence

$$\mathcal{D}'^{\overline{N}_1} \xrightarrow{.\mathcal{L}} \mathcal{D}'^{\overline{N}} \xrightarrow{\pi} M \longrightarrow 0. \tag{51}$$

The module M corresponds to the control system (44) in a sense that

$$\sum_{j=1}^{\overline{N}} \mathcal{L}_{ij}[E_j] = 0,\ 1 \le i \le \overline{N}_1, \tag{52}$$

that is, the system equations are always valid in M for $[E_j]$.

Remark 2. *In the case where the algebra $\mathcal{D}'$ is commutative we are also able to define a module $M' := \mathcal{D}'^{\overline{N}_1} / \mathcal{L}\mathcal{D}'^{\overline{N}}$. In this case we find that*

$$\mathcal{L}[D] = 0,\ \forall D \in M' \tag{53}$$

that is, the system equations are valid in M'. Furthermore, we have an exact sequence

$$\cdots \longrightarrow \mathcal{D}'^{\overline{N}} \xrightarrow{\mathcal{L}} \mathcal{D}'^{\overline{N}_1} \longrightarrow M' \longrightarrow 0 \tag{54}$$

A simple example of the commutative subalgebra is the first case in Example 7. The commutativity is very restrictive property for $\mathcal{D}'$. The module M' can be also defined in the case where $\mathcal{D}'$ is only an Ore algebra. *In this case (54) is valid. We do not consider module M' here.*

The module M is finitely generated because for every element
$[D] = [\begin{pmatrix} D_1 & \cdots & D_{\overline{N}} \end{pmatrix}] \in M$

$$[D] = \begin{pmatrix} D_1 & \cdots & D_{\overline{N}} \end{pmatrix} + \mathcal{D}'^{\overline{N}_1}\mathcal{L} = \sum_{j=1}^{\overline{N}} D_j [E_j]. \tag{55}$$

The elements $[E_j]$ do not, however, form necessarily a basis that is, every element $[D] \in M$ cannot be *uniquelly* expressed in the form $[D] = \sum_{j=1}^{\overline{N}} D_j [E_j]$. By (51) the module M is *finitely presented.*

A $\mathcal{D}'$-module M is *free* if it has a basis $[F_1], ..., [F_r]$. Free module is always $\mathcal{D}'$-isomorphic to the module $\mathcal{D}'^r$. A module M is *projective free* if there exists a $\mathcal{D}'$-module P such that the direct sum $M \oplus P = F$ is free. The *exact sequence* of projective $\mathcal{D}'$-modules P_j

$$\cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow 0 \tag{56}$$

is a *projective resolution* of M. The projective resolution always exists that is, we can choose projective $\mathcal{D}'$-modules $P_0,\ P_1,\ P_2, ...$ and $\mathcal{D}'$-homomorphisms $d_0,\ d_1,\ d_2, ...$ such that

$$\begin{aligned} d_0 : P_0 &\to M \text{ is onto} \\ d_1 : P_1 &\to \ker\, d_0 \text{ is onto} \\ d_2 : P_2 &\to \ker\, d_1 \text{ is onto} \end{aligned}$$

and so on.

3.3 Algebraic criterions for parametrizability

Let $\mathbf{S} : R^{N',m'} \to R^{N,m}$ be an operator as in section 3.1

$$\mathbf{S} = \begin{pmatrix} (\mathcal{S}'_{1jp}) & (\mathcal{S}'_{2jq}) \\ (\mathcal{S}'_{3kp}) & (\mathcal{S}'_{4kq}) \end{pmatrix} \in \mathcal{D}(N+m, N'+m') \tag{57}$$

Similarly to the construction of $\mathcal{L}$ define the matrix $\mathcal{S}$ corresponding to $\mathbf{S}$

$$\mathcal{S} = \begin{pmatrix} \mathcal{S}_{11} & \cdots & \mathcal{S}_{1\overline{N}'} \\ \cdots & \cdots & \cdots \\ \mathcal{S}_{\overline{N}1} & \cdots & \mathcal{S}_{\overline{N}\overline{N}'} \end{pmatrix} \in \mathcal{D}'(\overline{N} \times \overline{N}'). \tag{58}$$

The homomorphism $.\mathcal{S} : \mathcal{D}'^{\overline{N}} \to \mathcal{D}'^{\overline{N}'}$ is also defined as $.\mathcal{L}$.

Let A be a left $\mathcal{D}'$- module. In this algebraic setting we say, more generally, that the system $\mathcal{L}X = 0$, $X \in A^{\overline{N}}$, is $\mathcal{D}'$*-parametrizable in* $A^{\overline{N}}$ if there exists a matrix $\mathcal{S}$ given by (58) such that

$$\mathcal{L}X = 0 \Leftrightarrow X = (X_1, ..., X_{\overline{N}}) = \mathcal{S}X' \text{ for } X' \in A^{\overline{N}'}. \tag{59}$$

In our case A may be $\mathcal{R}$ or $\mathcal{D}'$, for example. When $A = \mathcal{R}$ we get the parametrization concept in the sense of subsection 3.1.

Denote $\mathrm{Hom}(P, A) = \{d|\ d : P \to A \text{ is } \mathcal{D}' - \text{homomorphism}\}$ for $\mathcal{D}'$-modules P and A. $\mathrm{Hom}(P, A)$ is a group with respect to addition but since $\mathcal{D}'$ is not necessarily commutative $\mathrm{Hom}(P, A)$ is not generally a $\mathcal{D}'$-module. We find that the mapping $\mathrm{Hom}(.\mathcal{L}, A) : \mathrm{Hom}(\mathcal{D}'^{\overline{N}}, A) \to \mathrm{Hom}(\mathcal{D}'^{\overline{N}_1}, A)$ defined by

$$\mathrm{Hom}(.\mathcal{L}, A)\phi = \phi \circ .\mathcal{L} \tag{60}$$

is a $\mathcal{D}'$-homomorphism.

Lemma 1. *The homomorphism* Hom$(.\mathcal{L}, A)$ *can be calculated by*

$$[\mathrm{Hom}(.\mathcal{L}, A)\phi](D) = \sum_{i=1}^{\overline{N}_1} D_i \left(\sum_{j=1}^{\overline{N}} \mathcal{L}_{ij}\phi(E_j) \right) \tag{61}$$

for $D = (D_1, ..., D_{\overline{N}_1}) \in \mathcal{D}'^{\overline{N}_1}$.

Proof. For any $\overline{D} = \sum_{j=1}^{\overline{N}} \overline{D}_j E_j \in \mathcal{D}'^{\overline{N}}$ and for any $\phi \in \mathrm{Hom}(\mathcal{D}'^{\overline{N}}, A)$ one has the expression

$$\phi(\overline{D}) = \sum_{j=1}^{\overline{N}} \overline{D}_j \phi(E_j)$$

and so

$$\begin{aligned} [\mathrm{Hom}(.\mathcal{L}, A)\phi](D) &= (\phi \circ .\mathcal{L})(D) = \phi(.\mathcal{L}D) = \phi(D\mathcal{L}) && (62) \\ &= \sum_{j=1}^{\overline{N}} (D\mathcal{L})_j \phi(E_j) = \sum_{i=1}^{\overline{N}_1} D_i \left(\sum_{j=1}^{\overline{N}} \mathcal{L}_{ij}\phi(E_j) \right) && (63) \end{aligned}$$

for $D \in \mathcal{D}'^{\overline{N}_1}$ as desired. □

Theorem 1. *The parametrizability condition*

$$\mathcal{L}X = 0 \Leftrightarrow X = \mathcal{S}X' \text{ for } X' \in A^{\overline{N}'}. \tag{64}$$

is equivalent to the equality

$$\ker \mathrm{Hom}(.\mathcal{L}, A) = \mathrm{im}\ \mathrm{Hom}(.\mathcal{S}, A). \tag{65}$$

Proof. A. Suppose that (64) holds. Then by Lemma 1 and by (64) we find that

$$\begin{aligned} &\phi \in \ker \mathrm{Hom}(.\mathcal{L}, A) \Leftrightarrow \mathrm{Hom}(.\mathcal{L}, A)\phi = 0 \Leftrightarrow \sum_{j=1}^{\overline{N}} \mathcal{L}_{ij}\phi(E_j) = 0 \\ &\Leftrightarrow \begin{pmatrix} \phi(E_1) \\ \vdots \\ \phi(E_{\overline{N}}) \end{pmatrix} = \mathcal{S}X' \Leftrightarrow \phi(E_j) = \sum_{k=1}^{\overline{N}'} \mathcal{S}_{jk}X'_k \\ &\Leftrightarrow \phi(D) = \sum_{j=1}^{\overline{N}} D_j \phi(E_j) = \sum_{j=1}^{\overline{N}} D_j \sum_{k=1}^{\overline{N}'} \mathcal{S}_{jk}X'_k. \end{aligned} \tag{66}$$

Define $\psi_{X'} \in \mathrm{Hom}(\mathcal{D}'^{\overline{N}'}, A)$ by $\psi_{X'}(\tilde{D}) := \sum_{k=1}^{\overline{N}'} \tilde{D}_k X'_k$. Then we have $X'_k = \psi_{X'}(E_k)$ and so by Lemma 1

$$\phi(D) = \sum_{j=1}^{\overline{N}} D_j \sum_{k=1}^{\overline{N}'} \mathcal{S}_{jk}\psi_{X'}(E_k) = [\mathrm{Hom}(.\mathcal{S}, A)\psi_{X'}](D) \Leftrightarrow \phi = \mathrm{Hom}(.\mathcal{S}, A)\psi_{X'}. \tag{67}$$

Since for any $\psi \in \mathrm{Hom}(\mathcal{D}^{\overline{N}'}, A)$

$$\psi = \psi_{X'}, \text{ for } X' = \begin{pmatrix} \psi(E_1) \\ \vdots \\ \psi(E_{\overline{N}'}) \end{pmatrix} \tag{68}$$

the first part of the assertion is proved.

B. Conversely, suppose that (65) holds. Define $\phi_X(D) = \sum_{j=1}^{\overline{N}} D_j X_j$. Then by Lemma 1

$$\mathcal{L}X = 0 \Leftrightarrow \sum_{j=1}^{\overline{N}} \mathcal{L}_{ij} X_j = 0,\ \forall i \Leftrightarrow \sum_{j=1}^{\overline{N}} \mathcal{L}_{ij} \phi_X(E_j) = 0,\ \forall i \tag{69}$$

$$\Leftrightarrow \sum_{i=1}^{\overline{N}_1} D_i \sum_{j=1}^{\overline{N}} \mathcal{L}_{ij} \phi_X(E_j) = 0,\ \forall D \in \mathcal{D}'^{\overline{N}_1} \Leftrightarrow \mathrm{Hom}(.\mathcal{L}, A)\phi_X = 0.$$

By (65) the equivalence (69) is valid if and only if

$$\phi_X = \mathrm{Hom}(.\mathcal{S}, A)\psi \Leftrightarrow \phi_X(D) = [\mathrm{Hom}(.\mathcal{S}, A)\psi](D) = \sum_{j=1}^{\overline{N}} D_j \sum_{k=1}^{\overline{N}'} \mathcal{S}_{jk} \psi(E_k),\ \forall D \in \mathcal{D}'^{\overline{N}}$$

$$\Leftrightarrow X_j = \sum_{k=1}^{\overline{N}'} \mathcal{S}_{jk} \psi(E_k) \Leftrightarrow X = \mathcal{S}X',\ \text{for } X' = \begin{pmatrix} \psi(E_1) \\ \vdots \\ \psi(E_{\overline{N}'}) \end{pmatrix}. \tag{70}$$

Due to (68) this completes the proof. □

The homomorphism $.\mathcal{S} : \mathcal{D}'^{\overline{N}} \to \mathcal{D}'^{\overline{N}'}$ is similarly defined as $.\mathcal{L}$. Let $\tilde{N}$ be the $\mathcal{D}'$-module

$$\tilde{N} = \mathcal{D}'^{\overline{N}'} / \mathcal{D}'^{\overline{N}} \mathcal{S}. \tag{71}$$

For the definition and for the basic properties of the homology groups $\mathrm{Ext}^n(M, A)$ and $\mathrm{Tor}^n(M, A)$ (see e.g. [22, 10, 17])

We have

Theorem 2. *Suppose that there exists a matrix (58) such that*

$$D\mathcal{L} = D' \Leftrightarrow D'\mathcal{S} = 0. \tag{72}$$

Then the parametrizability condition (65) is valid if and only if

$$\mathrm{Ext}^1(\tilde{N}, A) = 0. \tag{73}$$

Proof. One sees that the assumption (72) is equivalent to

$$.\mathcal{L}D = D' \Leftrightarrow .\mathcal{S}D' = 0. \tag{74}$$

Hence assumption (72) is equivalent to the exactness of the following sequence

$$\cdots \longrightarrow \mathcal{D}'^{\overline{N}_1} \xrightarrow{.\mathcal{L}} \mathcal{D}'^{\overline{N}} \xrightarrow{.\mathcal{S}} \mathcal{D}'^{\overline{N}'} \xrightarrow{\tilde{\pi}} \tilde{N} \longrightarrow 0. \tag{75}$$

The resolution (75) implies the truncated dual sequence

$$0 \longrightarrow \mathrm{Hom}(\mathcal{D}'^{\overline{N}'}, A) \xrightarrow{\mathrm{Hom}(.\mathcal{S},A)} \mathrm{Hom}(\mathcal{D}'^{\overline{N}}, A) \xrightarrow{\mathrm{Hom}(.\mathcal{L},A)} \mathrm{Hom}(\mathcal{D}'^{\overline{N}_1}, A) \longrightarrow \cdots \tag{76}$$

$\mathrm{Ext}^1(\tilde{N}, A) := \ker \mathrm{Hom}(.\mathcal{L}, A)/\mathrm{im}\ \mathrm{Hom}(.\mathcal{S}, A) = 0$ if and only if the condition (65) is valid. Hence Theorem 1 implies the assertion. □

The condition (73) can also be studied without any explicit calculations. For example, since the module $\tilde{N}$ is finitely presented, the condition (73) is valid when the module $\tilde{N}$ is flat. In the case where A is injective $\mathrm{Ext}^1(\tilde{N}, A) = 0$. Especially, $\mathrm{Ext}^1(\tilde{N}, A) = 0$ in the case where $\tilde{N}$ is projective. Note that neither $\mathcal{D}'$ nor $\mathcal{R}$ are generally injective modules.

The conditions like (73) are interesting because they are independent of the used (projective) free resolutions. So they give valuable information for the instrinsic structure of the system.

Example 9. *Let $G =]0,1[$ and Δ is any open interval of $\mathbf{R}$. Consider the system*

$$\begin{aligned} \frac{\partial^2 v_1}{\partial t^2} &= \frac{\partial^2 v_1}{\partial x^2} + v_2 \\ v_1(0,t) &= v_1(1,t) = 0. \end{aligned}$$

In practice v_1 may be the state variable and v_2 may be the control variable. We have $N = 2,\ N_1 = 1,\ m_1 = 1$ and $\partial G = \{0,1\}$. For notational convenience we may assume that $m = 1$ although no boundary value functions w_1 exist. Hence $\overline{N} = 2,\ \overline{N}_1 = 1$. Denote $\partial_1 = \frac{\partial}{\partial x},\ \partial_2 = \frac{\partial}{\partial t}$.

We see that

$$\mathcal{L} := \begin{pmatrix} \mathcal{L}_1 & \mathcal{L}_2 \end{pmatrix} = \left(\begin{pmatrix} r^+(\partial_2^2 - \partial_1^2) & 0 \\ r'I & 0 \end{pmatrix} \quad \begin{pmatrix} -I & 0 \\ 0 & 0 \end{pmatrix} \right). \tag{77}$$

We seek a possible operator $\mathcal{S}$ for which the assumption (72) holds. Let $D = \begin{pmatrix} \mathcal{A}_1 & \mathcal{A}_2 \\ \mathcal{A}_3 & \mathcal{A}_4 \end{pmatrix} \in \mathcal{D}$ and $(D', D'') \in \mathcal{D}^2$ where $D' = \begin{pmatrix} \mathcal{A}_1' & \mathcal{A}_2' \\ \mathcal{A}_3' & \mathcal{A}_4' \end{pmatrix}$, $D'' = \begin{pmatrix} \mathcal{A}_1'' & \mathcal{A}_2'' \\ \mathcal{A}_3'' & \mathcal{A}_4'' \end{pmatrix}$. We find that

$$D\mathcal{L} = D' \tag{78}$$

if and only if

$$\left(\begin{pmatrix} \mathcal{A}_1 r^+(\partial_2^2 - \partial_1^2) + \mathcal{A}_2 r'I & 0 \\ \mathcal{A}_3 r^+(\partial_2^2 - \partial_1^2) + \mathcal{A}_4 r'I & 0 \end{pmatrix} \quad \begin{pmatrix} -\mathcal{A}_1 & 0 \\ -\mathcal{A}_3 & 0 \end{pmatrix} \right) = \left(\begin{pmatrix} \mathcal{A}_1' & \mathcal{A}_2' \\ \mathcal{A}_3' & \mathcal{A}_4' \end{pmatrix} \quad \begin{pmatrix} \mathcal{A}_1'' & \mathcal{A}_2'' \\ \mathcal{A}_3'' & \mathcal{A}_4'' \end{pmatrix} \right) \tag{79}$$

if and only if

$$\begin{aligned} &\mathcal{A}_2'' = \mathcal{A}_4'' = \mathcal{A}_2' = \mathcal{A}_4' = 0,\ -\mathcal{A}_1 = \mathcal{A}_1'',\ -\mathcal{A}_3 = \mathcal{A}_3'' \\ &-\mathcal{A}_1'' r^+(\partial_2^2 - \partial_1^2) + \mathcal{A}_2 r'I = \mathcal{A}_1',\ -\mathcal{A}_3'' r^+(\partial_2^2 - \partial_1^2) + \mathcal{A}_4 r'I = \mathcal{A}_3. \end{aligned} \tag{80}$$

Let K be the potential operator

$$Kg(x,t) = xg(1,t) + (1-x)g(0,t). \tag{81}$$

Then we observe that $r'Kg = g$ or $r'K = I'$. Multiplying the equation

$$-\mathcal{A}_1'' r^+(\partial_2^2 - \partial_1^2) + \mathcal{A}_2 r'I = \mathcal{A}_1' \tag{82}$$

by K we get

$$\mathcal{A}_2 = \mathcal{A}_1' K + \mathcal{A}_1'' r^+(\partial_2^2 - \partial_1^2)K. \tag{83}$$

Similarly we get from the equation

$$-\mathcal{A}_3'' r^+(\partial_2^2 - \partial_1^2) + \mathcal{A}_4 r'I = \mathcal{A}_3' \tag{84}$$

that

$$\mathcal{A}_4 = \mathcal{A}_3' K + \mathcal{A}_3'' r^+(\partial_2^2 - \partial_1^2)K. \tag{85}$$

Hence by (80)

$$-\mathcal{A}_1'' r^+(\partial_2^2 - \partial_1^2) + (\mathcal{A}_1' K + \mathcal{A}_1'' r^+(\partial_2^2 - \partial_1^2)K) r'I = \mathcal{A}_1' \tag{86}$$

and

$$-\mathcal{A}_3''r^+(\partial_2^2-\partial_1^2)+(\mathcal{A}_3'K+\mathcal{A}_3''r^+(\partial_2^2-\partial_1^2)K)r'I=\mathcal{A}_3'. \tag{87}$$

Expressing the equations (86), (87) and

$$\mathcal{A}_2''=\mathcal{A}_4''=\mathcal{A}_2'=\mathcal{A}_4'=0 \tag{88}$$

in the matrix form we have

$$\left(\begin{pmatrix}\mathcal{A}_1' & \mathcal{A}_2'\\ \mathcal{A}_3' & \mathcal{A}_4'\end{pmatrix}\quad\begin{pmatrix}\mathcal{A}_1'' & \mathcal{A}_2''\\ \mathcal{A}_3'' & \mathcal{A}_4''\end{pmatrix}\right)\mathcal{S}=0 \tag{89}$$

where

$$\mathcal{S}=\begin{pmatrix}\begin{pmatrix}I_{1,0}-Kr'I & 0\\ 0 & I\end{pmatrix} & \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\\ \begin{pmatrix}(r^+(\partial_2^2-\partial_1^2)(-I+Kr'I) & 0\\ 0 & 0\end{pmatrix} & \begin{pmatrix}0 & 0\\ 0 & I'\end{pmatrix}\end{pmatrix}. \tag{90}$$

In section 3.5 we shall find that $\mathcal{S}$ is the $\mathcal{D}'$-parametrization in any module A^2.

3.4 Application of adjoints

The adjoints can be applied also in the algebraic analysis which we describe in the sequel. In this subsection we *assume that the required formal adjoints exist in the subalgebra $\mathcal{D}'$.*

Remark 3. *If necessary we can apply order reducing to quarantee the existence of formal adjoints. For example, in the case where $A=\mathcal{R}$ this can be described as follows.*

Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are isomorphisms $\mathcal{R}^{\overline{N}}\to\mathcal{R}^{\overline{N}}$ and $\mathcal{R}^{\overline{N}_1}\to\mathcal{R}^{\overline{N}_1}$, respectively. Then the control system $\mathcal{L}u=0$ is equivalent to the system

$$(\mathcal{T}'\circ\mathcal{L}\circ\mathcal{T})u'=0 \tag{91}$$

where $\mathcal{T}u'=u$. Especially, the operators $\mathcal{T}$ and $\mathcal{T}'$ can be chosen to be order reducing that is, the orders and/or classes of the composed operators are less than the original operators (section 2.2.3). In parametrization we are able to use order reducing operators to quarantee the existence of adjoints. This is based on the following observation

$$\mathcal{L}u=0\Leftrightarrow(\mathcal{T}'\circ\mathcal{L}\circ\mathcal{T})u'=0\Leftrightarrow u'=\mathcal{S}'f\Leftrightarrow u=(\mathcal{T}\circ\mathcal{S}')f\Leftrightarrow u=\mathcal{S}f \tag{92}$$

where $\mathcal{S}'$ is the parametrization of the system $(\mathcal{T}'\circ\mathcal{L}\circ\mathcal{T})u'=0$ and $\mathcal{S}=\mathcal{T}\circ\mathcal{S}'$.

Let $\mathcal{L}^*$ be the formal adjoint of $\mathcal{L}$

$$\mathcal{L}^*=\begin{pmatrix}\mathcal{L}_{11}^* & \cdots & \mathcal{L}_{\overline{N}_1 1}^*\\ \cdots & \cdots & \cdots\\ \mathcal{L}_{1\overline{N}}^* & \cdots & \mathcal{L}_{\overline{N}_1\overline{N}}^*\end{pmatrix}\in\mathcal{D}'(\overline{N}_1\times\overline{N}) \tag{93}$$

defined as in 2.2.2. The homomorphism $.\mathcal{L}^*:\mathcal{D}'^{\overline{N}}\to\mathcal{D}'^{\overline{N}_1}$ is similarly defined as $.\mathcal{L}$. Let N be the $\mathcal{D}'$-module

$$N=\mathcal{D}'^{\overline{N}_1}/\mathcal{D}'^{\overline{N}}\mathcal{L}^*. \tag{94}$$

Suppose that N has a *free resolution with finitely generated modules*

$$\cdots\longrightarrow\mathcal{D}'^{\overline{N}'}\xrightarrow{.\mathcal{S}^*}\mathcal{D}'^{\overline{N}}\xrightarrow{.\mathcal{L}^*}\mathcal{D}'^{\overline{N}_1}\xrightarrow{\pi^*}N\longrightarrow 0. \tag{95}$$

A $\mathcal{D}'$-module E is *stably free* if one can find a finitely generated free module F such that $E\oplus F$ is a finitely generated free module that is, $E\oplus F$ is isomorphic to $\mathcal{D}'^n$ for some n. A projective stably free module N always admits the free resolution (95) with finitely generated modules, for

example ([11]). In addition, in the case where $\mathcal{D}'$ is Noetherian N has the free resolution with finitely generated modules (95). For example, the subalgebra

$$\mathcal{D}' = \{\begin{pmatrix} r^+A(D) & 0 \\ 0 & 0 \end{pmatrix} \mid A(D) \text{ is a PDO with constant coefficients}\}.$$

is a commutative Noetherian integral domain. Besides this kind of simple cases, substantial further work must be done to analyze algebraic properties of more general systems.

Remark 4. Suppose that $A = \mathcal{D}'$. *Since* $\mathcal{L}D = 0 \Leftrightarrow D = \mathcal{S}D'$ *is equivalent to*

$$D^*\mathcal{L}^* = 0 \Leftrightarrow D^* = D'^*\mathcal{S}^* \tag{96}$$

the parametrizability condition (59) in the case $A = \mathcal{D}'$ *is equivalent to the existence of free resolution (95).*

Suppose that the free resolution (95) exists. Let $\mathcal{S} := \mathcal{S}^{**}$. As above we see that the condition $\mathrm{Ext}^1(N, \mathcal{D}') = 0$ is equivalent to compatibility condition

$$\mathcal{L}^*\tilde{D} = \tilde{D}' \Leftrightarrow \mathcal{S}^*\tilde{D}' = 0. \tag{97}$$

Let $D = \tilde{D}^*,\ D' = \tilde{D}'^*$. Using the relation $(\mathcal{L}^*\tilde{D})^* = D\mathcal{L} = .\mathcal{L}D$ and similarly $(\mathcal{S}^*\tilde{D}')^* = .\mathcal{S}D'$ one sees that the compatibility condition (97) is equivalent to

$$.\mathcal{L}D = D' \Leftrightarrow .\mathcal{S}D' = 0. \tag{98}$$

Hence the condition

$$\mathrm{Ext}^1(N, \mathcal{D}') = 0 \tag{99}$$

is equivalent to the assumption (72) of Theorem 2.

Example 10. *A. Let* $n = 2$. *Denote* $\partial_1 = \frac{\partial}{\partial x_1},\ \partial_2 = \frac{\partial}{\partial x_2}$. *We choose*

$$\mathcal{D}' = \left\{\begin{pmatrix} P(\partial_1, \partial_2) & 0 \\ 0 & 0 \end{pmatrix} \mid P(\partial_1, \partial_2) \ is\ PDO\ with\ constant\ coefficients\right\}. \tag{100}$$

Consider the following system (without boundary values)

$$\partial_1 v_2 - \partial_2 v_1 = 0. \tag{101}$$

In this case we have

$$\mathcal{L} = \begin{pmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \end{pmatrix},\ u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},\ u_1 = \begin{pmatrix} v_1 \\ w_1 \end{pmatrix},\ u_2 = \begin{pmatrix} v_2 \\ w_2 \end{pmatrix} \tag{102}$$

where $\mathcal{L}_{11} = \begin{pmatrix} \partial_1 & 0 \\ 0 & 0 \end{pmatrix},\ \mathcal{L}_{12} = \begin{pmatrix} -\partial_2 & 0 \\ 0 & 0 \end{pmatrix}$. *We find that* $\overline{N} = 2,\ \overline{N}_1 = 1$.

B. The formal adjoint exists and it is

$$\mathcal{L}^* = \begin{pmatrix} \mathcal{L}^*_{11} \\ \mathcal{L}^*_{12} \end{pmatrix} \tag{103}$$

where $\mathcal{L}^*_{11} = \begin{pmatrix} -\partial_1 & 0 \\ 0 & 0 \end{pmatrix},\ \mathcal{L}^*_{12} = \begin{pmatrix} \partial_2 & 0 \\ 0 & 0 \end{pmatrix}$. *The module* N *is given by*

$$N = \mathcal{D}'/\mathcal{D}'^2\mathcal{L}^*. \tag{104}$$

Consider the existence of the resolution (95). Let $(D', D'') \in \mathcal{D}'^2$*. Denote* $D' = \begin{pmatrix} P' & 0 \\ 0 & 0 \end{pmatrix}$, $D'' = \begin{pmatrix} P'' & 0 \\ 0 & 0 \end{pmatrix}$ *where* $P' = P'(\partial_1, \partial_2),\ P'' = P''(\partial_1, \partial_2)$*. We find that*

$$.\mathcal{L}^*(D', D'') = 0 \Leftrightarrow \begin{pmatrix} D' & D'' \end{pmatrix} \mathcal{L}^* = 0 \tag{105}$$

if and only if

$$\begin{pmatrix} P' & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -\partial_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} P'' & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \partial_2 & 0 \\ 0 & 0 \end{pmatrix} = 0 \Leftrightarrow -P'\partial_1 + P''\partial_2 = 0.$$

Furthermore, the symbol of $-P'\partial_1 + P''\partial_2$ *is the polynomial* $-P'(\xi_1\xi_2)\xi_1 + P''(\xi_1, \xi_2)\xi_2$ *and one knows from algebra that there exists a polynomial* Q *such that ([11], p. 113)*

$$-P'(\xi_1\xi_2)\xi_1 + P''(\xi_1, \xi_2)\xi_2 = 0 \Leftrightarrow P'(\xi_1\xi_2) = Q(\xi_1, \xi_2)\xi_2,\ P''(\xi_1, \xi_2) = Q(\xi_1, \xi_2)\xi_1. \tag{106}$$

Hence the condition $-P'\partial_1 + P''\partial_2 = 0$ *is equivalent that*

$$P' = Q(\partial_1, \partial_2)\partial_2,\ P'' = Q(\partial_1, \partial_2)\partial_1. \tag{107}$$

Denote $\mathcal{S}^* = \left(\begin{pmatrix} \partial_2 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} \partial_1 & 0 \\ 0 & 0 \end{pmatrix} \right)$*. As a conclusion we see that*

$$\begin{aligned} &.\mathcal{L}^*(D', D'') = 0 \Leftrightarrow (D', D'') = Q(\partial_1, \partial_2) \left(\begin{pmatrix} \partial_2 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} \partial_1 & 0 \\ 0 & 0 \end{pmatrix} \right) = Q(\partial_1, \partial_2)\mathcal{S}^* \\ &\Leftrightarrow (D', D'') = .\mathcal{S}^* Q. \end{aligned} \tag{108}$$

Hence $\overline{N}' = 1$ *and the free resolution (95) with finitely generated modules exists.*

C. It is easy to see that $\mathrm{Ext}^1(N, \mathcal{D}') = 0$*. Furthermore,*

$$\mathcal{S} = \mathcal{S}^{**} = - \begin{pmatrix} \begin{pmatrix} \partial_2 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} \partial_1 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}. \tag{109}$$

$\mathrm{Ext}^1(\tilde{N}, \mathcal{R}) = 0$ *if and only if*

$$\mathcal{S}U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \Leftrightarrow \mathcal{L} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0 \tag{110}$$

where $U = \begin{pmatrix} V_1 \\ W_1 \end{pmatrix} \in \mathcal{R}$ *and* $u_1 = \begin{pmatrix} v_1 \\ w_1 \end{pmatrix}$, $u_2 = \begin{pmatrix} v_2 \\ w_2 \end{pmatrix} \in \mathcal{R}$*. Hence we see that* $\mathrm{Ext}^1(\tilde{N}, \mathcal{R}) = 0$ *if and only if*

$$\nabla V_1 = (v_1, v_2) \Leftrightarrow \partial_1 v_2 - \partial_2 v_1 = 0 \tag{111}$$

which is valid if the set G *is convex, for example.*

3.5 Projectivity

We finally shortly treat the projectiveness of the underlying modules. The module M is projective if and only if $\mathrm{Ext}^1(M, A) = 0$ for any left module A. The *projective dimension* is defined by

$$\mathrm{P} - \dim\ M = \inf_{n \geq 0} \{\mathrm{Ext}^{n+1}(M, A) = 0 \text{ for any } A\}. \tag{112}$$

We see that P-dim $M = 0$ if and only if M is projective. For non-projective modules M projective dimension measures how far M is from being projective.

The following theorem gives a tool to show the projectiveness in our case.

Theorem 3. *The module M is projective if and only if there exists a $\mathcal{D}'$-homomorphism $.\mathcal{P}$: $\mathcal{D}'^{\overline{N}} \to \mathcal{D}'^{\overline{N}_1}$ (so called* lift*) such that*

$$.\mathcal{L} \circ .\mathcal{P} \circ .\mathcal{L} = .\mathcal{L}. \tag{113}$$

The proof of theorem can be found e.g. in [20] . The condition (113) is equivalent to the existence of a matrix $\mathcal{P} \in \mathcal{D}'(\overline{N} \times \overline{N}_1)$ such that

$$\mathcal{L}\mathcal{P}\mathcal{L} = \mathcal{L}. \tag{114}$$

Corollary 1. *Suppose that there exists a $\mathcal{D}'$-homomorphism $.\mathcal{P}$: $\mathcal{D}'^{\overline{N}_1} \to \mathcal{D}'^{\overline{N}}$ (so called* left inverse*) such that*

$$.\mathcal{P} \circ .\mathcal{L} = \mathcal{I}_{\overline{N}_1} \tag{115}$$

where $\mathcal{I}_{\overline{N}_1}$ is the identity in $\mathcal{D}'^{\overline{N}_1}$. Then the module M is projective.

The condition (115) is equivalent to the existence of a matrix $\mathcal{P} \in \mathcal{D}'(\overline{N} \times \overline{N}_1)$ such that

$$\mathcal{L}\mathcal{P} = \mathcal{I}_{\overline{N}_1} := \begin{pmatrix} \mathcal{I} & 0 \cdots 0 & & \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \mathcal{I} \end{pmatrix} \in \mathcal{D}'(\overline{N}_1 \times \overline{N}_1). \tag{116}$$

Projectivity gives the following sufficient condition for the parametrizability

Theorem 4. *Suppose that there exists a matrix $\mathcal{P} \in \mathcal{D}'(\overline{N} \times \overline{N}_1)$ such that*

$$\mathcal{L}\mathcal{P}\mathcal{L} = \mathcal{L}. \tag{117}$$

Then the control system $\mathcal{L}X = 0$ is $\mathcal{D}'$-parametrizable in $A^{\overline{N}}$.

Proof. From (117) we get

$$\mathcal{L}(\mathcal{I}_{\overline{N}} - \mathcal{P}\mathcal{L}) = 0. \tag{118}$$

Hence the operator $\mathcal{S} = \mathcal{I}_{\overline{N}} - \mathcal{P}\mathcal{L}$ satisfies the following condition: If $X = \mathcal{S}X'$ then $\mathcal{L}X = 0$. Conversely, suppose that $\mathcal{L}X = 0$. Then

$$X = (\mathcal{I}_{\overline{N}} - \mathcal{P}\mathcal{L})X + (\mathcal{P}\mathcal{L})X = \mathcal{S}X \tag{119}$$

which completes the proof. □

A special case of the Theorem 4 is

Corollary 2. *Suppose that the matrix $\mathcal{L}$ has a* right inverse *that is, there exists a matrix $\mathcal{P} \in \mathcal{D}(\overline{N} \times \overline{N}_1)$ such that*

$$\mathcal{L}\mathcal{P} = \mathcal{I}_{\overline{N}_1}. \tag{120}$$

Then the control system $\mathcal{L}X = 0$ is $\mathcal{D}'$-parametrizable in $A^{\overline{N}}$.

Under the assumptions of Theorem 4 the parametrization is given by

$$\mathcal{S}X' = (\mathcal{I}_{\overline{N}} - \mathcal{P}\mathcal{L})X', \ X' \in A^{\overline{N}}. \tag{121}$$

We find that the projectiveness is a structural property under which the parametrizability does not depend on the module A.

Remark 5. *A. The module M is* flat *if and only if Tor-functor is exact. Projective module is flat but there exist nonprojective flat modules. Flatness is a useful additional concept in the study of these system modules.*

B. For example, for integral domains $\mathcal{D}'$ one sees by a direct computation that (98) or (72) is equivalent to the torsion freeness *of M that is, $\tilde{D}[D] = 0$, $\tilde{D} \in \mathcal{D}'$, $[D] \in M$ if and only if $\tilde{D} = 0$ or $[D] = 0$. For nonintegral domains this need not be valid. Recall that*

$$\textit{free} \subset \textit{stably free} \subset \textit{projective} \subset \textit{flat} \subset \textit{torsion free}. \tag{122}$$

Example 11. *Reconsider the Example 9. Recall that*

$$\mathcal{L} := \begin{pmatrix} \mathcal{L}_1 & \mathcal{L}_2 \end{pmatrix} = \left(\begin{pmatrix} r^+(\partial_2^2 - \partial_1^2) & 0 \\ r'I & 0 \end{pmatrix} \quad \begin{pmatrix} -I & 0 \\ 0 & 0 \end{pmatrix} \right). \tag{123}$$

The corresponding complex related to the module M is

$$\longrightarrow \mathcal{D} \xrightarrow{.\mathcal{L}} \mathcal{D}^2 \xrightarrow{\pi} M \longrightarrow 0. \tag{124}$$

Furthermore, $D.\mathcal{L} = 0$ for $D = \begin{pmatrix} \mathcal{A}_1 & \mathcal{A}_2 \\ \mathcal{A}_3 & \mathcal{A}_4 \end{pmatrix} \in \mathcal{D}$ if and only if

$$\mathcal{A}_1 = \mathcal{A}_3 = \mathcal{A}_2 r'I = \mathcal{A}_4 r'I = 0. \tag{125}$$

The last two equations imply that also $\mathcal{A}_2 = \mathcal{A}_4 = 0$ and so $D = 0$. Hence we get the free resolution

$$0 \longrightarrow \mathcal{D} \longrightarrow \mathcal{D}^2 \xrightarrow{.\mathcal{L}} M \xrightarrow{\pi} 0. \tag{126}$$

The truncated dual sequence is

$$0 \longrightarrow \mathrm{Hom}(\mathcal{D}^2, \mathcal{D}) \xrightarrow{\Phi} \mathrm{Hom}(\mathcal{D}, \mathcal{D}) \longrightarrow 0 \tag{127}$$

where $\Phi(\phi) = \mathrm{Hom}(.\mathcal{L}, \mathcal{D})\phi = \phi \circ .\mathcal{L}$. The homomorphism Φ is surjective: Due to Lemma 1 we see that $\Phi(\phi) = \psi$ for $\psi \in \mathrm{Hom}(\mathcal{D}, \mathcal{D})$ if and only if

$$\sum_{j=1}^{2} \mathcal{L}_j \phi(E_j) = \psi(E_1). \tag{128}$$

Choose $\phi(D_1, D_2) = (D_1, D_2)\mathcal{P}\psi(E_1)$ where

$$\mathcal{P} = \begin{pmatrix} \begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} -I & r^+(\partial_2^2 - \partial_1^2)K \\ 0 & 0 \end{pmatrix} \end{pmatrix}.$$

Here, as above, K is the potential operator

$$Kg(x,t) = xg(1,t) + (1-x)g(0,t) \tag{129}$$

(observe that $r'Kg = g$). Then (128) holds and so Φ is surjective. Hence $\mathrm{Ext}^1(M, \mathcal{D}) = 0$.

Actually M is projective. The sequence (126) splits and the lift can be chosen to be $.\mathcal{P}$. We can compute the parametrization by (121) and the result is

$$\begin{aligned} \mathcal{S} &= \begin{pmatrix} \begin{pmatrix} I & 0 \\ 0 & I' \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} I & 0 \\ 0 & I' \end{pmatrix} \end{pmatrix} - \mathcal{P}\mathcal{L} \\ &= \begin{pmatrix} \begin{pmatrix} I - Kr'I & 0 \\ 0 & I' \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} (r^+(\partial_2^2 - \partial_1^2)(-I + Kr'I) & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & I' \end{pmatrix} \end{pmatrix}. \end{aligned} \tag{130}$$

By (130) the parametrization in the sense of the subsection 3.1 is given elementwise (that is in the case $A = \mathcal{R}$) by

$$\begin{aligned} v_1 &= f_1 - Kr'f_1 \\ v_2 &= r^+(\partial_2^2 - \partial_1^2)f_1 - r^+(\partial_2^2 - \partial_1^2)Kr'f_1, \ f_1 \in C^\infty(\overline{G} \times \Delta) \\ w_1 &= g_1, \ g_1 \in C^\infty(\partial G \times \Delta). \end{aligned} \tag{131}$$

Note that $Kr'f_1 = xf_1(1,\cdot) + (1-x)f_1(0,\cdot)$. The last equation of (131) is due to the notational covensions and it is superflous.

Example 12. *Let $n = 2$ and let $\Delta v_1 = \frac{\partial^2 v_1}{\partial x_1^2} + \frac{\partial^2 v_1}{\partial x_2^2}$. Consider the system*

$$(1-\Delta)v_1 = 0 \tag{132}$$
$$\frac{\partial v_1}{\partial \nu}|_{\partial G} = w_1. \tag{133}$$

Here we have $N = m = N_1 = m_1 = 1$ and so $\overline{N} = \overline{N}_1 = 1$. In addition,

$$\mathcal{L} = \begin{pmatrix} 1-\Delta & 0 \\ r'\frac{\partial}{\partial \nu} & -I' \end{pmatrix} \tag{134}$$

where I' is the identity operator on $C^\infty(\partial G)$. Note that $r'\frac{\partial}{\partial \nu} = r'(\nu_1 \frac{\partial}{\partial x_1} + \nu_2 \frac{\partial}{\partial x_2})$. We seek a matrix $\mathcal{P} \in \mathcal{D}$ such that

$$\mathcal{L}\mathcal{P} = \mathcal{I}_1 = \begin{pmatrix} I & 0 \\ 0 & I' \end{pmatrix}. \tag{135}$$

Let $\mathcal{P} = \begin{pmatrix} \mathcal{P}_1 & \mathcal{P}_2 \\ \mathcal{P}_3 & \mathcal{P}_4 \end{pmatrix}$. We find that the condition (135) is equivalent to

$$(1-\Delta)\mathcal{P}_1 = I,\ (1-\Delta)\mathcal{P}_2 = 0,\ r'\frac{\partial}{\partial \nu}\mathcal{P}_1 - \mathcal{P}_3 = 0,\ r'\frac{\partial}{\partial \nu}\mathcal{P}_2 - \mathcal{P}_4 = I'. \tag{136}$$

We can choose $\mathcal{P}_2 = 0$ and then $\mathcal{P}_4 = -I'$. Let $\mathcal{P}_1 = r^+A$ where r^+A is the pseudo-differential operator with symbol $1/(1+||\xi||^2)$. Then $(1-\Delta)\mathcal{P}_1 = I$. From $r'\frac{\partial}{\partial \nu}\mathcal{P}_1 - \mathcal{P}_3 = 0$ we get $\mathcal{P}_3 = r'\frac{\partial}{\partial \nu}\mathcal{P}_1$. Hence we can choose

$$\mathcal{P} = \begin{pmatrix} r^+A & 0 \\ r'\frac{\partial}{\partial \nu}(r^+A) & -I' \end{pmatrix}. \tag{137}$$

The corresponding parametrization is

$$\mathcal{S} = \mathcal{I}_1 - \mathcal{P}\mathcal{L} = \begin{pmatrix} I & 0 \\ 0 & I' \end{pmatrix} - \begin{pmatrix} r^+A & 0 \\ r'\frac{\partial}{\partial \nu}(r^+A) & -I' \end{pmatrix}\begin{pmatrix} 1-\Delta & 0 \\ r'\frac{\partial}{\partial \nu} & -I' \end{pmatrix} \tag{138}$$
$$= \begin{pmatrix} I-(r^+A)(1-\Delta) & 0 \\ -r'\frac{\partial}{\partial \nu}(r^+A)(1-\Delta) + r'\frac{\partial}{\partial \nu} & 0 \end{pmatrix}. \tag{139}$$

Elementwise the parametrization is given by

$$v_1 = (I-(r^+A)(1-\Delta))f_1 \tag{140}$$
$$w_1 = -r'\frac{\partial}{\partial \nu}(r^+A)(1-\Delta)f_1 + r'\frac{\partial f_1}{\partial \nu}.$$

The matrix $\mathcal{P}$ can be chosen several other ways. For example, we can choose $\mathcal{P}_1$ such that $(1-\Delta)\mathcal{P}_1 = I,\ r'\frac{\partial}{\partial \nu}\mathcal{P}_1 = 0$. Then $\mathcal{P}_1 = r^+A + B$ where B is a singular Green operator ([8]). Choosing $\mathcal{P}_2 = 0$ we get

$$\mathcal{P} = \begin{pmatrix} r^+A + B & 0 \\ 0 & -I' \end{pmatrix}. \tag{141}$$

Parametrization can be computed and it is of the form

$$v_1 = (I-(r^+A+B)(1-\Delta))f_1 \tag{142}$$
$$w_1 = r'\frac{\partial f_1}{\partial \nu}.$$

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